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Factorizable non-atomic copulas

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ABSTRACT

Generalizing the notion of invariant sets by Darsow and Olsen, Sumetkijakan studied a subclass of singular copulas, the so-called non-atomic copulas, defined via its associated σ-algebras. It was shown that the Markov operator of every non-atomic copula is partially factorizable, i.e. it is the composition of left and right invertible Markov operators on a subspace of $L^1([0, 1])$ depending on the copula. Here, we further investigate the associated σ-algebras of the product of certain copulas and obtain (1) a sharper result on the partial factorizability of non-atomic copulas and (2) the existence and uniqueness of a completely factorizable copula that shares the same set of associated σ-algebras as that of a given non-atomic copula.

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1. Introduction

One of the most notable developments in the study of copulas is due to Darsow et al. (1992), where a binary operation $*$ on the set of bivariate copulas was introduced and thoroughly studied. It was shown by Olsen et al. (1996) that there is a bijective isomorphism between the set of copulas equipped with the $*$-product and the set of Markov operators equipped with the composition. In proving a characterization of idempotent copulas, Darsow and Olsen (2010) introduced the notion of *invariant sets*, forming a σ-algebra, of the corresponding Markov operator. Recently (Sumetkijakan, 2017), the notion was extended to two associated σ-algebras and non-atomic copulas were defined and proved to be partially factorizable. The class of non-atomic copulas contains all non-atomic idempotent copulas and is a subclass of the singular copulas.

Simple examples of non-atomic copulas (D_0 and D_1 in Section 2.3) clearly shows that the associated σ-algebras σ_C and σ_C^* cannot identify the support of a non-atomic copula C. Nonetheless, we prove that for such a copula C, there correspond a left invertible copula L and a right invertible copula R such that $\sigma_L^* = \sigma_C^*$, $\sigma_R = \sigma_C$ and C can be partially factorized as the $*$-product of L and R, when viewed through their corresponding Markov operators. This sharpens and improves Theorem 4.6 in Sumetkijakan (2017). However, it does not necessarily mean that $C = L * R$. Even in the case that it does, the σ-algebras σ_C and σ_C^* cannot determine the support of C as there are many left invertible copulas L with the same σ-algebra σ_L^* and likewise for right invertible copulas R. Finally, although the pair (L, R) is far from unique, we prove that for a given non-atomic copula C, the factorizable copula $L * R$ sharing the same set of associated σ-algebras is in fact unique. As a consequence, a factorizability criterion is obtained.

This article is organized as follows. Section 2 provides essential backgrounds and terminologies required in this manuscript. Section 2.1 lays out basic knowledge in copulas and Markov operators while Section 2.2 lists some notions and

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quoted results related to non-atomic copulas. Section 2.3 demonstrates how to compute the associated σ-algebras of a few motivated examples. Section 3 contains results concerning the associated σ-algebras of the product of certain non-atomic copulas, and summarizes in a sharper theorem on the partial factorizability of non-atomic Markov operators. Section 4 introduces factorizable copulas and shows that every non-atomic copula has a unique factorizable copula that shares the same set of associated σ-algebras. As a result, a factorizability test of non-atomic copulas is proposed. To make the key ideas more transparent, the final section gives a summary of the main results of the article.

2. Background knowledge

Throughout the manuscript, let $\mathscr{B} \equiv \mathscr{B}(\mathbb{I})$ and $\mathscr{B}(\mathbb{I}^2)$ denote the Borel σ-algebra on $\mathbb{I} \equiv [0, 1]$ and $\mathbb{I}^2$, respectively, λ Lebesgue measure on $\mathbb{I}$, and $\mathbb{1}_A$ the indicator function of a Borel set $A \in \mathscr{B}$. Given a σ-algebra $\mathscr{S} \subseteq \mathscr{B}$, the class of integrable $\mathscr{S}$-measurable functions on $\mathbb{I}$ is denoted by $L^1(\mathbb{I}, \mathscr{S}, \lambda)$, or $L^1(\mathbb{I}, \mathscr{S})$ for short, and $L^1(\mathbb{I}) \equiv L^1(\mathbb{I}, \mathscr{B})$.

2.1. Copulas and Markov operators

A function $C : \mathbb{I}^2 \to \mathbb{I}$ is said to be a copula if it fulfills the following three conditions for all $u, v, u', v' \in \mathbb{I}$: (i) $C(u, 0) = C(0, v) = 0$, (ii) $C(u, 1) = u$ and $C(1, v) = v$, and (iii) $C(u', v') - C(u', v) - C(u, v') + C(u, v) \geq 0$ whenever $u \leq u'$ and $v \leq v'$. The most well-known examples of copulas are the independence copula $\Pi(u, v) = uv$ and the Fréchet–Hoeffding upper and lower bounds $M(u, v) = \min\{u, v\}$ and $W(u, v) = \max\{0, u + v - 1\}$. Every copula C induces a unique doubly stochastic measure μ_C on $(\mathbb{I}^2, \mathscr{B}(\mathbb{I}^2))$ defined by $\mu_C((a, b] \times (c, d]) = C(b, d) - C(b, c) - C(a, d) + C(a, c)$. This measure-theoretic connection leads to a natural notion of the *support* of a copula C as the support of its induced measure μ_C. The *transpose* of a copula C is defined as $C^t(u, v) = C(v, u)$ for $u, v \in \mathbb{I}$, and C is said to be *symmetric* if $C^t = C$. The product of copulas C and D is defined by $(C * D)(x, y) = \int_0^1 \partial_2 C(x, t)\, \partial_1 D(t, y)\, \mathrm{d}t$ for $x, y \in \mathbb{I}$. It can be proved that $C * D$ is indeed a copula and that the $*$-product is associative and distributive over convex combinations of copulas. Recall that the class of copulas is convex. A copula C is said to be *left (right) invertible* if there is a copula D in which $D * C = M$ ($C * D = M$). If C is both left and right invertible, we say that C is *invertible*. Denote by $\mathcal{F}$ the set of Borel *measure-preserving transformations* of the interval $\mathbb{I}$, that is, Borel functions f satisfying $\lambda(f^{-1}(B)) = \lambda(B)$ for all $B \in \mathscr{B}$. Define the copula C_{fg} induced by f and g in $\mathcal{F}$ by

$$C_{fg}(u, v) = \lambda\left(f^{-1}[0, u] \cap g^{-1}[0, v]\right) \quad \text{for } u, v \in \mathbb{I}.$$

For any $f \in \mathcal{F}$, C_{ef} is left invertible with $C_{ef}^t = C_{fe}$ as its left inverse where e denotes the identity map on $\mathbb{I}$. A function $f \in \mathcal{F}$ is said to possess an *essential inverse* $g \in \mathcal{F}$ if $g \circ f = e = f \circ g$ almost everywhere. Denote by $\mathcal{F}_{\text{inv}}$ the set of measure-preserving functions that possess essential inverses.

A linear operator $T : L^1(\mathbb{I}) \to L^1(\mathbb{I})$ is called a *Markov operator* if:

(M1) $f \geq 0$ implies $Tf \geq 0$ for all $f \in L^1(\mathbb{I})$,
(M2) $T\mathbb{1}_{\mathbb{I}} = \mathbb{1}_{\mathbb{I}}$, and
(M3) $\int_{\mathbb{I}} Tf \, \mathrm{d}\lambda = \int_{\mathbb{I}} f \, \mathrm{d}\lambda$ for all $f \in L^1(\mathbb{I})$.

By standard arguments (Olsen et al., 1996), any Markov operator T is a bounded operator on $\mathbb{I}$ with $\|T\| = 1$. Denote by $\mathcal{C}$ the set of copulas and $\mathcal{M}$ the set of Markov operators. Olsen et al. (1996) provided a one-to-one correspondence between the space $\mathcal{C}$ equipped with the $*$-product and the space $\mathcal{M}$ equipped with the composition operator $\circ$ via the isomorphisms $C \mapsto T_C$ and $T \mapsto C_T$ defined by

$$(T_C\psi)(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_0^1 \partial_2 C(x, t)\psi(t)\, \mathrm{d}t \quad \text{for } x \in \mathbb{I}$$

and $C_T(u, v) = \int_0^u (T\mathbb{1}_{[0,v]})(s)\, \mathrm{d}s$ for $u, v \in \mathbb{I}$. In fact, $T_{C*D} = T_C \circ T_D$. Some well-known examples of Markov operators induced by copulas are $T_M\psi(x) = \psi(x)$, $T_W\psi(x) = \psi(1 - x)$, and $T_{\frac{M+W}{2}} = \frac{1}{2}[T_M + T_W]$. To avoid double subscripting, the Markov operator induced by C_{fg} will be written as T_{fg}. Denote by T^* the adjoint operator of T. Even though T^* is originally defined on $L^\infty(\mathbb{I})$, it has a unique extension to a Markov operator on $L^1(\mathbb{I})$, cf. Sumetkijakan (2017) and Printechapat (2017). It is evident from Olsen et al. (1996) that $T_C^* = T_{C^t}$, hence $T_C^{**} = T_C$. See Durante and Sempi (2015) and Nelsen (2006) for detailed introduction to copulas.

2.2. Non-atomic copulas and associated σ-algebras

Let $\mathscr{S}$ and $\mathscr{R}$ be sub-σ-algebras of $\mathscr{B}$. $\mathscr{S}$ is said to be *essentially equivalent to* $\mathscr{R}$ if for each $S \in \mathscr{S}$ there exists $R \in \mathscr{R}$ such that $\lambda(S \triangle R) = 0$ where $S \triangle R = (S \setminus R) \cup (R \setminus S)$. It is said that $\mathscr{S}$ and $\mathscr{R}$ are *essentially equivalent*, written $\mathscr{S} = \mathscr{R}$ *essentially*, or $\mathscr{S} \approx \mathscr{R}$ for short, if $\mathscr{S}$ is essentially equivalent to $\mathscr{R}$ and vice versa. A set $S \in \mathscr{S}$ is called an *atom* in $\mathscr{S}$ if (i) $\lambda(S) > 0$ and (ii) for each $E \in \mathscr{S}$ either $\lambda(S \cap E) = \lambda(S)$ or $\lambda(S \cap E) = 0$. If there is no atom in $\mathscr{S}$, then $\mathscr{S}$ is called a *non-atomic* σ-algebra; otherwise it is called *atomic*. Recall (Sumetkijakan, 2017) that the *associated σ-algebras* of a copula C are defined as

$$\sigma_C = \{S \in \mathscr{B} \mid \exists R \in \mathscr{B},\ T_C\mathbb{1}_S = \mathbb{1}_R\} \quad \text{and} \quad \sigma_C^* = \{R \in \mathscr{B} \mid \exists S \in \mathscr{B},\ T_C\mathbb{1}_S = \mathbb{1}_R\}.$$

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