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Mean square stability for systems on stochastically generated discrete time scales

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ABSTRACT

We consider dynamic systems which evolve on discrete time domains where the time steps form a sequence of independent, identically distributed random variables. In particular, we classify the mean-square stability of linear systems on these time domains using quadratic Lyapunov functionals. In the case where the system matrix is a function of the time step, our results agree with and generalize stability results found in the Markov jump linear systems literature. In the case where the system matrix is constant, our results generalize, illuminate, and extend to the stochastic realm results in the field of dynamic equations on time scales. In order to help see the factors that contribute to stability, we prove a sufficient condition for the solvability of the Lyapunov equation by appealing to a fixed point theorem of Ran and Reurings. Finally, an example using observer-based feedback control is presented to demonstrate the utility of the results to control engineers who cannot guarantee uniform timing of the system.

1. Introduction

The theory of dynamic equations on time scales was first introduced the seminal paper [1]. This work aimed to unify various overarching concepts from the (sometimes disparate) theories of discrete and continuous dynamical systems, as in [2], but also to extend these theories to more general classes of dynamical systems. From there, time scales theory advanced fairly quickly, culminating in the introductory text [3] and the more advanced monograph [4].

In recent years, the theory of time scales has been gaining attention in the literature in the analysis of control systems subject to non-uniform timing, as might occur in the case of communication dropout in a control network [5–8]. The Kalman filter on time scales has been studied [9], as has the Luenberger observer [10,11], state feedback control [12–15], and observability, controllability, and reachability [16–20]. The stability for linear and quasilinear systems on an arbitrary time scale is of particular interest, and has been well studied [5,6,10,21–32].

In the case of linear time invariant systems, Pötzsche et al. [29] showed that the eigenvalues of the system matrix existing in a certain region of the complex plane was necessary and sufficient for exponential stability under the assumption of uniform regressivity of the time scale. Additionally, they showed how to calculate the region of exponential stability. The paper also raised two issues which are still guiding current research. The first issue is, unlike the cases where $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = h\mathbb{Z}, h > 0$, on a general time scale the region of exponential stability is not equivalent to the other varieties of stability [33]. The second issue is that the region of exponential stability is in general difficult to compute. To get around the

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computational difficulties of the region of exponential stability, Gard and Hoffacker [30] found that the region of exponential stability always contained a disc tangent to the origin with a curvature equal to the largest distance between points in the time scale. Oftentimes, this region is a very conservative estimate of the stability region.

The stability of dynamic equations with general structured perturbations [34], nonlinear finite-dimensional control systems [35], and switched systems [6,31,32,36–39] on time scales have also been studied.

In much of the time scales literature, the time scale is deterministic and fixed. In applications, it may be more realistic that the timing disruptions are stochastic in nature. In [40], the authors classify exponential stability almost surely for linear time invariant systems on so-called purely discrete stochastic time scales, that is, a random set where the time steps form a sequence of independent, identically distributed random variables (and hence each realization of the random set is a time scale). In [41], the authors showed that the existence of a stochastic quadratic Lyapunov function implied that the zero solution was asymptotically stable in the sense of Kushner, and, in the linear time invariant case, gave conditions on the eigenvalues of the system matrix for there to be a stochastic quadratic Lyapunov function.

Adding a stochastic structure to the time scale allows us to draw on and make connections with the results in the stochastic systems stability literature. A linear system on a purely-discrete stochastic time scale, for example, is a Markov chain. Similarly, a linear control system on a purely-discrete stochastic time scale is a Markov jump linear system. Stability in both of these settings has been studied extensively, as collected in the excellent references [42,43] for Markov chains and [44] for Markov jump linear systems.

The main objective of the current work is to bridge and unify stability results in the stochastic systems literature and stability results in the time scales literature. In particular, we aim to prove theorems which allow us to determine the stability of a dynamic equation on a stochastic time scale given the probability distribution of the time steps. In this paper, we show that the existence of a stochastic quadratic Lyapunov function actually classifies mean-square stability of solutions linear dynamic equations on time scales where the system matrix is a function of the time step. One consequence of this is the emergence of a matrix Lyapunov equation which has the form of a perturbed Stein equation [45]. The existence of a solution to this matrix Lyapunov equation is equivalent to the existence of a quadratic Lyapunov function, demonstrating that these equations arise naturally in the study of stochastic systems. Moreover, we extend on the work of [11] to checkable conditions for the stability of a Luenberger observer on a stochastic time scale. Viewed in the correct way, our main result is a generalization of a result in the Markov jump linear systems literature [44], but cast in terms of the statistics of the time steps.

2. Preliminaries

For simplicity, we present the scalar version of the stability theory on time scales. Before introducing stochastic time scales, we give some background on deterministic time scales. A time scale, $\mathbb{T}$, is any closed subset of the real line. The *forward jump operator* [3], $\sigma(t)$, is defined as the point immediately to the right of t, in the sense that $\sigma(t) = \inf\{s \in \mathbb{T} \mid s > t\}$. The *graininess* is the distance between points defined as $\mu(t) := \sigma(t) - t$. For $\mathbb{R}$, $\sigma(t) = t$ and $\mu(t) = 0$.

The *time scale* or *Hilger derivative* of a function $x(t)$ on $\mathbb{T}$ is defined as

$$x^{\Delta}(t) := \frac{x(\sigma(t)) - x(t)}{\mu(t)}. \tag{1}$$

On $\mathbb{R}$, this is interpreted in the limit as $\mu \to 0^+$ and $x^{\Delta}(t) = \frac{d}{dt}x(t)$. The *Hilger integral* can be viewed as the antiderivative in the sense that, if $y(t) = x^{\Delta}(t)$, then for $s, t \in \mathbb{T}$,

$$\int_{\tau=s}^{t} y(\tau)\Delta\tau = x(t) - x(s).$$

In [3], it is shown that the standard problem

$$x^{\Delta}(t) = \lambda(t)x(t), \quad x(t_0) = x_0, \tag{2}$$

has a unique forward solution denoted by $x(t) = x_0 e_{\lambda}(t, t_0)$ where

$$e_{\lambda}(t, s) := \exp\left(\int_{\tau=s}^{t} \frac{\operatorname{Log}(1 + \mu(\tau)\lambda(\tau))}{\mu(\tau)}\Delta\tau\right).$$

For each $t \in \mathbb{T}$, define the *Hilger Circle* by

$$\mathcal{H}_{\mu(t)} := \{z \in \mathbb{C} : |1 + z\mu(t)| < 1, z \neq -1/\mu(t)\} \tag{3}$$

Note that $\mathcal{H}_{\mu(t)}$ is a disc of radius $1/\mu(t)$ contained in the left half-plane tangent to the imaginary axis. We note that the term Hilger circle is a misnomer, but has become common in the literature.

For a more thorough treatment of time scales, see the text by Bohner and Peterson [3]. Extending the ideas of standard or deterministic time scales, we consider stochastic time scales next.