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# Existence of optimal controls on hybrid time domains

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ABSTRACT

Hybrid control systems are considered, combining continuous-time dynamics and discrete-time dynamics, and modeled by differential equations or inclusions, by difference equations or inclusions, and by constraints on the resulting dynamics. Solutions are defined on hybrid time domains. Finite-horizon and infinite-horizon optimal control problems for such control systems are considered. Existence of optimal open-loop controls is shown. The assumptions used include, essentially, the existence for the (non-hybrid) continuous-time case; the existence for the (non-hybrid) discrete-time case; mild conditions on the endpoint penalties; and closedness and boundedness, in the finite-horizon case, of the set of admissible hybrid time domains. Examples involving switching systems and hybrid automata are included.



## 1. Introduction

As stated in [1], "Hybrid control systems are control systems that involve both continuous and discrete dynamics and continuous and discrete controls". Building on the hybrid systems framework of [2], hybrid control systems were modeled a combination of differential equations or inclusions with input, difference equations or inclusions with input, and constraints on states and inputs that determine where the continuous and the discrete dynamics apply, for example in [3]. These modeling tools allow for treatment of hybrid control systems with explicit discrete variables, also known as hybrid automata, and of classes of switching systems. For a thorough discussion, see [2] or [4]. In this framework, solutions are considered on hybrid time domains, also known as hybrid time trajectories [5] or hybrid time sets [6], and so are parameterized by time $t$ and the number of jumps $j$. This allows for multiple discrete transitions, or jumps, at a single time instant and the framework includes continuous-time and discrete-time systems as special cases. Optimal control has been applied to systems in this framework, for example by [7–9], and [10]. This note provides existence of optimal open-loop controls for both finite-horizon and infinite-horizon quite general optimal control problems.

Optimal control for hybrid systems parameterized by time $t$ only, and often explicitly mentioning discrete variables, or logical modes, and the closely-related impulsive differential equations, has seen a more extensive treatment. Optimality conditions, generalizing the maximum principle to the hybrid setting, appear in [11–15], and more. For numerical methods, see [16] and the references therein. Early general existence result are in [1], which also includes an extensive discussion of previous work and related frameworks, [17], and [18] for the stochastic case. These results use strict assumptions including one on time separation between jumps. Similar assumptions appear in [19], on a quasi-variational inequality describing the optimal value function. Existence is assumed in works like [16,20,21], or [22], sometimes in addition to upper bounds on the number of switches/jumps [21] assumed for other purposes, and further conditions on switching surfaces/guards [16] that ensure that the underlying set of time domains is compact. Related work on switching systems, like [23] or [24–26] does not provide general existence results, while [27] suggests that without bounds on the number of switches, optimal solutions

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need not exist. Conditions bounding the number of switches/jumps also appear in general, not necessarily optimal, hybrid control design [28].

Hybrid optimal control problems in this note explicitly involve a fixed set of hybrid time domains, and look for an optimal process with its domain in that set. For the finite-horizon case, it is assumed that the set of hybrid time domains is bounded and closed with respect to set convergence, and this assumption is discussed in some detail in the sequel and illustrated by examples. The main existence result can be summarized as stating that if this assumption holds, and –essentially –if existence holds when the continuous-time dynamics are considered separately and when the discrete-time dynamics are considered separately, then optimal hybrid controls exist for the hybrid optimal control problem. The proof is direct: it uses a minimizing sequence to construct an optimal process. Results are, in a sense, expected, since if a limiting hybrid time domain exists for hybrid time domains of a minimizing sequence, then standard non-hybrid results imply that an optimal process on that limiting hybrid time domain can be deduced, but do not appear to have been written in this generality.

A different approach to the existence of optimal processes, possibly with several jumps at the same time, is in [29], where it is deduced from the existence of a solution to a quasi-variational inequality describing the optimal value function. In [29], in contrast to this work, there is no state constraints, or constraints on the time domains, though the considered costs bound the number of jumps of the considered time interval. In time-scale literature, see [30] and the references therein, a time scale can model several consecutive jumps, and much more exotic behaviors, but the time scale is always fixed a priori. Similarly, the hybrid time domain is fixed a priori in, for example, [8] and [10]. The set-up and the approach taken here was used, for particular costs and for the infinite-time horizon only, in [31].

## 2. Hybrid inclusions with input

A hybrid inclusion with input is represented by

$$\begin{array}{ll} (x,u)\in C & \dot{x}\in F(x,u) \\ (x,u)\in D & x^{+}\in G(x,u)\,, \end{array} \tag{1}$$

where $C, D \subset \mathbb{R}^{n+m}$ are sets and $F, G : \mathbb{R}^{n+m} \rightrightarrows \mathbb{R}^n$ are set-valued mappings. A nonempty set $E \subset \mathbb{R}^2$ is a *compact hybrid time domain* if $E$ has the form

$$\bigcup_{j=j_a}^{j_b} [t_j, t_{j+1}] \times \{j\}, \tag{2}$$

where $j_a \le j_b$ are integers and $t_j$ are real numbers so that $t_{j_a} \le t_{j_a+1} \le \cdots \le t_{j_b} \le t_{j_b+1}$. A set $E$ is a *hybrid time domain* if, for every $(t_a, j_a), (t_b, j_b) \in E$ with $t_a \le t_b, j_a \le j_b$, the set $E_{(t_a,j_a)}^{(t_b,j_b)} := \{(t,j) \in E \mid t_a \le t \le t_b, j_a \le j \le j_b\}$ is a compact hybrid time domain. For a hybrid time domain $E$,

$$\sup_t E := \sup\{t \mid \exists j \in \mathbb{Z}\, (t,j) \in E\}, \quad \sup_j E := \sup\{j \mid \exists t \in \mathbb{R}\, (t,j) \in E\},$$

$\sup E := (\sup_t E, \sup_j E)$, and $\inf_t E, \inf_j E, \inf E$ are defined similarly. It is said that $\inf E$ is finite if both $\inf_t E, \inf_j E$ are finite.

An *admissible input* or *an open-loop control* is a function $u : \operatorname{dom} u \to \mathbb{R}^m$, so that the domain $\operatorname{dom} u$ of $u$ is a hybrid time domain and, if

$$I_j(u) := \{t \mid (t,j) \in \operatorname{dom} u\} = \operatorname{dom} u \cap (\mathbb{R} \times \{j\})$$

has nonempty interior, denoted $\operatorname{int} I_j(u)$, then $t \mapsto u(t,j)$ is locally integrable. Note the special role of $u(\tau, j)$, where $\tau$ is the right endpoint of $I_j$, below: $u(\tau, j)$ does not affect the flow dynamics but only the jump dynamics.

Given an admissible input $u : \operatorname{dom} u \to \mathbb{R}^m$, a *solution to the hybrid system with input* (1) resulting from $u$ is a function $\phi : \operatorname{dom}\phi \to \mathbb{R}^n$ such that

- $\operatorname{dom}\phi = \operatorname{dom} u$;
- if $\inf\operatorname{dom} u$ is finite and $\inf\operatorname{dom} u \in \operatorname{dom} u$, then $(\phi(\inf\operatorname{dom} u), u(\inf\operatorname{dom} u)) \in \overline{C} \cup D$;
- if $I_j(\phi)$ has nonempty interior, then $t \mapsto \phi(t,j)$ is locally absolutely continuous on $I_j(\phi)$ and
$$(\phi(t,j), u(t,j)) \in C \text{ for all } t \in \operatorname{int} I_j(\phi) \text{ and}$$
$$\frac{d}{dt}\phi(t,j) \in F(\phi(t,j), u(t,j)) \quad \text{for almost all } t \in I_j(\phi);$$
- if $(t,j) \in \operatorname{dom}\phi$ and $(t,j+1) \in \operatorname{dom}\phi$ then
$$(\phi(t,j), u(t,j)) \in D \text{ and } \phi(t,j+1) \in G(\phi(t,j), u(t,j)).$$

A pair $(u, \phi)$, where $u$ is an admissible input and $\phi$ is a resulting solution to (1) will be referred to as a *process*. The notation $\operatorname{dom}(u,\phi)$ represents the hybrid time domain of the process $(u,\phi)$, which equals $\operatorname{dom} u$ and $\operatorname{dom}\phi$. A *forward complete process* is then a process $(u,\phi)$ so that $\sup\operatorname{dom}(u,\phi)$ is not finite. A *compact process* is a process $(u,\phi)$ so that $\operatorname{dom}(u,\phi)$ is compact.