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Fractional derivatives with no-index law property: Application to chaos and statistics

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ABSTRACT

Recently fractional differential operators with non-index law properties have being recognized to have brought new weapons to accurately model real world problems particularly those with non-Markovian processes. This present paper has two double aims, the first was to prove the inadequacy and failure of index law fractional calculus and secondly to show the application of fractional differential operators with no index law properties to statistic and dynamical systems. To achieve this, we presented the historical construction of the concept of fractional differential operators from Leibniz to date. Using a matrix based on the fractional differential operators, we proved that, fractional operators obeying index law cannot model real world problems taking place in two states, more precisely they cannot describe phenomena taking place beyond their boundaries, as they are scaling invariant, more precisely our results show that, mathematical models based on these differential operators are not able to describe the inverse memory, meaning the full history of a physical problem cannot be described accurately using these derivatives with index law properties. On the other hand, we proved that, differential operators with no index-law properties are scaling variant, thus can describe situations taking place in different states and are able to localize the frontiers between two states. We present the renewal process properties included in differential equation build out of the Atangana–Baleanu fractional derivative and counting process, which is connected to its inter-arrival time distribution Mittag–Leffler distribution which is the kernel of these derivatives. We presented the connection of each derivative to a statistical family, for instance Riemann–Liouville–Caputo derivatives are connected to the Pareto statistic, which has no well-defined average when alpha is less than 1 corresponding to the interval where fractional operators mostly defined. We established new properties and theorem for the Atangana–Baleanu derivative of an analytic function, in particular we proved that, they are convolution of the Mittag–Leffler function with the Riemann–Liouville–Caputo derivatives. To see the accuracy of the non-index law derivative to in modeling real chaotic problems, 4 examples were considered, including the nine-term 3-D novel chaotic system, King Cobra chaotic system, the Ikeda delay system and chaotic chameleon system. The numerical simulations show very interesting and novel attractors. The king cobra system with the Atangana–Baleanu presented a very novel attractor where at the earlier time we observed a random walk and latter time we observed the real sharp of the cobra. The Ikeda model with Atangana–Baleanu presented different attractors for each value of fractional order, in particular we obtain a square and circular explosions. The results obtained in this paper show that, the future of modeling real world problem relies on fractional differential operators with non-index law property. Our numerical results showed that, to not model physical problems with fractional differential operators with non-singular kernel and imposing index law in fractional calculus is rightfully living with closed eyes without ever taking a risk to open them.

1. Introduction

In the last decade, the concept of fractional differentiation and integration or particularly the concept of non-local operators has attracted from many fields of science, technology and engineering, from almost all parts of our globe attention of humankind. Due

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to their abilities of including into mathematical models some observed real world behaviors, the concept has, recruited even engineers and some were converted to applied mathematicians. Due to some challenges faced by researchers to capture more natural observed facts, several definitions where suggested [1–6], we quote only the most used in the last two years, Caputo–power law, the Caputo–Fabrizio and the Atangana–Baleanu fractional derivatives. From a discussion of Leibniz and L'Hôpital, the idea of creating a derivative with fractional order was born. Now let us recall the history of fractional calculus, just after the question asked by L'Hôpital to Leibniz in 1675, 15 years later, he initiated a fractional derivative of the function exponential by reiterating the differentiation the following function [7]:

$$\frac{d^n e^{\beta x}}{dx^n} = \beta^n e^{\beta x}, \qquad n = 1, 2, 3. \tag{1}$$

Now he decided to replace the n by a fractional v to have

$$\frac{d^v e^{\beta x}}{dx^v} = \beta^v e^{\beta x}, \qquad n = 1, 2, 3. \tag{2}$$

If now one look at the connection of derivative in the Fourier space, we have the following relationship

$$\hat{f}^{(v)}(x) = (-ix)^v \hat{f}(x). \tag{3}$$

With the above relations in mind, Liouville in his turn in 1832 considered a function $f(x)$ with the following representation:

$$f(x) = \sum_{j=1}^{\infty} a_x e^{\beta_j x}. \tag{4}$$

Then he applied the Leibniz formulation or the v-derivative of the exponential function and obtained in his turn the following equation:

$$\frac{d^v}{dx^v} f(x) = \sum_{j=1}^{\infty} a_x \beta_j^v e^{\beta_j x}. \tag{5}$$

They both believed that the fractional integral must be formulated with the series that converges of course in fact they also both believed that, a fractional differential operator could be obtained using an exponential function as this function has several properties can be observed in real world. Eq. (5) was then at that time considered as the fundamental construction of a fractional derivative by also using their Laplace transforms. Now later on with the ambition to obtain a fix formulation of fractional derivative, Liouville suggested another formula but this time based on integral as presented in below formula:

$$J^\lambda(t) = \int_0^\infty e^{-xt} t^{\lambda-1} dt, \qquad \lambda > 0, \qquad x > 0. \tag{6}$$

Nevertheless, by a change of variable it was very easier to obtain the following:

$$x^{-\lambda} = \frac{1}{\Gamma(\lambda)} J(x). \tag{7}$$

Now if one considered the fractional derivative in Liouville sense with order v, the following formulation is obtained [7]:

$$\frac{d^v}{dx^v} x^{-\lambda} = \frac{1}{\Gamma(\lambda)} \int_0^\infty \frac{d^v}{dx^v} e^{-xt} t^{\lambda-1} dt,$$

$$= \frac{(-1)^v}{\Gamma(\lambda)} \int_0^\infty e^{-xt} t^{\lambda+v-1} dt = (-1) \frac{\Gamma(\lambda+v)}{\Gamma(\lambda)} x^{-\lambda-v}, \qquad \lambda, v > 0. \tag{8}$$

Later on, about 1876 Riemann provided the following fractional derivative of a given function $f(x)$ [7]:

$$f^{(v)}(x) = \int_a^x (x-\tau)^{-v-1} f(\tau) d\tau + \Psi(x), \qquad v < 0. \tag{9}$$

Liouville call the additional function a complementary function, as he believed that, the classical differential equation, mean that differential equation based on the concept of the rate of change setting

$$\frac{dy^n}{dx^n} = 0, \Rightarrow y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots + C_{n-1} x^{n-1}. \tag{10}$$

Then after combination of ideas, the Riemann–Liouville fractional derivative was born from the fractional integral.

The Riemann–Liouville has been in use for several centuries. We can even say that, for more than 300 years no researcher suggested criteria except Ross [8]. Until recently, some authors suggested some criteria to classify fractional differential operators. The idea of providing a guideline in the field was not at all a bad, however the list of items suggested introduced a restriction and the critics raised were not academically acceptable. Due to these criticisms, several authors investigated the list and their results rejected the index-law, among these researchers, Nieto et al., wrote a paper with title “It is possible to construct a fractional derivative such that index law holds?” their results disproved the inclusion of index law in the field [9]. Angel et al. did another throughout investigation of the diffusive role of some kernel [10] and also their results suggested that only operators with non-index law properties have crossover diffusive behaviors. Caputo and Fabrizio wrote a very detailed research work in which they proved that, the suggested index law was wrong or rather was a limitation to the field and in their turn they suggested a list of items to be followed see [11]. In this same paper, the authors proved the need of non-singular differential operators and their application to nature. Independently, Djida et al., presented an optimal control of diffusion with the Atangana–Baleanu fractional differential operators and surprisingly they proved that the existence of solution were obtained for all values of fractional order meaning $\alpha \in (0, 1)$, and they pointed out that, the existence of the solution with Riemann–Liouville and Caputo were obtained only when $\alpha \in (0, 0.5)$, also this results are perfectly connected to those presented in [12]. Atangana investigated evolution equations associated to each derivative to check the validity of index law and their C_0 semigroup properties, surprisingly, the results revealed that, although the Riemann–Liouville differential operator obeys the index law its associate evolution equation does not due of course of the memory effect. While the Caputo–Fabrizio fractional derivative does not obey index-law in [13], but its associate evolution equation satisfied all the principles of the strong continuous semi-group. Nevertheless, in general the Atangana–Baleanu fractional differential operator does satisfy the index law but only when the fractional order are the same; also its associate evolution equation does not consistently also due to memory effect. Using physical observed fact and the history of evolution of physic it was demonstrated in a paper with title “Decolonization of fractional calculus rules: breaking commutativity and associativity to capture more natural phenomena” [14] that the index-law is not welcome in the field of fractional differential and integral operators.

2. Index-law and its limitations

Let us take time in this section and recall readers that are not acquainted with the concept of index law that come from commutativity. What is really commutativity, what is the physical interpretation of this mathematical concept that is coming from classical mechanic? What advantages we do have when an operator is commutative or does obey the index law? What are the disadvantages also? What are the associate geometry of commutativity and what is that of non-commutativity? What are their connections in real world problems? We shall start this section then, by recalling the mathematical meaning of commutativity. We recognized that