Statistics and Probability Letters 83 (2013) 2499–2506

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Absolute continuity of the laws of a multi-dimensional stochastic differential equation with coefficients dependent on the maximum

Tomonori Nakatsu *

Department of Mathematical Science, Ritsumeikan University, 1-1-1 Nojihigashi, Kusatsu, Shiga, 525-8577, Japan
Japan Science and Technology Agency, Japan

ARTICLE INFO

Article history:
Received 11 June 2013
Received in revised form 9 July 2013
Accepted 21 July 2013
Available online 26 July 2013

MSC:
60H07
60H10

Keywords:
Absolutely continuous law
Stochastic differential equation
Malliavin calculus

ABSTRACT

In this article, we consider an m-dimensional stochastic differential equation with coefficients which depend on the maximum of the solution. First, we prove the absolute continuity of the law of the solution. Then we prove that the joint law of the maximum of the ith component of the solution and the i'th component of the solution is absolutely continuous with respect to the Lebesgue measure in a particular case. The main tool to prove the absolute continuity of the laws is Malliavin calculus.

1. Introduction and preliminaries of Malliavin calculus

In this article, we deal with the following m-dimensional stochastic differential equation (SDE):

$$X_t^i = x^i + \sum_{l=1}^{d} \int_0^t A_l^i(s, X_s, M_s) dW_s^l + \int_0^t B^i(s, X_s, M_s) ds, \quad 1 \le i \le m, \tag{1}$$

where W denotes a d-dimensional Brownian motion, $A_l^i, B^i : [0, \infty) \times \mathbb{R}^{2m} \to \mathbb{R}$, $1 \le i \le m$, $1 \le l \le d$ are measurable functions, and $M_s = (M_s^1, \dots, M_s^m) := (\max_{u \le s} X_u^1, \dots, \max_{u \le s} X_u^m)$. The purpose of this article is to prove the absolute continuity of the joint laws concerning the solution to (1) with Lipschitz continuous coefficients by using Malliavin calculus. In Fournier and Printems (2010), the authors proved that, if $m = d = 1$, A and B are Hölder continuous, then for $t > 0$, the law of X_t is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, where X is a weak or strong solution to (1). They used the characteristic function of X_t to prove the absolute continuity of the law of X_t. As far as we know, their technique cannot be extended to the multi-dimensional case.

In this article, first we prove the absolute continuity of the law of $X_t = (X_t^1, \dots, X_t^m)$ with respect to the Lebesgue measure on $\mathbb{R}^m$. Then we prove that, for $1 \le i, i' \le m$, the law of $(M_t^i, X_t^{i'})$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2$ when each A_l^i in (1) does not depend on the second space variable. To deal with the Malliavin

* Correspondence to: Department of Mathematical Science, Ritsumeikan University, 1-1-1 Nojihigashi, Kusatsu, Shiga, 525-8577, Japan. Tel.: +81 09078791592.
E-mail address: nakatu.tomonori@gmail.com.

0167-7152/$ – see front matter
http://dx.doi.org/10.1016/j.spl.2013.07.011

derivative of M_t^i, some delicate arguments stated as Lemmas 2 and 4 in this article are needed, and to study the time when $\{X_s^i, s \in [0,t]\}$ attains its maximum on $[0,t]$ is important for the proof of the absolute continuity of the law of $(M_t^i, X_t^{i'})$. To analyze the law of $(M_t^i, X_t^{i'})$ may be important in applications such as finance (see Fournié et al. (1999) or Chapter 6 of Nualart (2006)).

We introduce some basic tools of Malliavin calculus that will be used throughout the article. We refer to Nualart (2006). Let $(\Omega, \mathcal{F}, P)$ be the canonical Wiener space which supports a d-dimensional Brownian motion W. The class of real random variables of the form $F = f(W_{t_1}, \dots, W_{t_n}), f \in C_b^\infty(\mathbb{R}^{nd}; \mathbb{R}), 0 \le t_1, \dots, t_n \le t$ is denoted by $\mathcal{S}$. $\mathbb{D}^{1,p}$ denotes a Banach space which is the completion of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p} = E[|F|^p]^{\frac{1}{p}} + \left(E\left[\left(\int_0^t \sum_{j=1}^d |D_r^j F|^2 dr \right)^{\frac{p}{2}} \right] \right)^{\frac{1}{p}},$$

where

$$D_r^j F := \sum_{i=1}^n \frac{\partial f}{\partial x_{ji}} (W_{t_1}, \dots, W_{t_n}) \mathbf{1}_{[0,t_i]}(r).$$

$\mathbb{D}^{k,p}$ is defined analogously, and its associated norm is denoted by $\| \cdot \|_{k,p}$. Also, we define $\mathbb{D}^{k,\infty} = \cap_{p \ge 1} \mathbb{D}^{k,p}$ and $\mathbb{D}^\infty = \cap_{p\ge 1} \cap_{k \ge 1} \mathbb{D}^{k,p}$. For $F \in \mathbb{D}^{1,2}$, we define $\|DF\|_H^2 := \int_0^t \sum_{j=1}^d |D_r^j F|^2 dr$.

Now let us introduce a localization of $\mathbb{D}^{k,p}$. $\mathbb{D}_{loc}^{k,p}$ denotes the set of random variables F such that there exists a sequence $\{(\Omega_n, F_n), n \ge 1\} \subset \mathcal{F} \times \mathbb{D}^{k,p}$ with the following properties.

(i) $\Omega_n \uparrow \Omega$, a.s.
(ii) $F = F_n$, a.s. on Ω_n.

2. The existence, uniqueness, and differentiability of solutions to (1) and the absolute continuity of the probability law of X_t

In this section, first we prove the existence, uniqueness, and differentiability of solutions to (1). Then we prove that for $t > 0$ the law of X_t is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^m$ where X is the solution to (1).

We assume the following.

(A1) There exist $K, L > 0$ such that

$$|A(t,x_1,x_2) - A(t,x_1',x_2')| + |B(t,x_1,x_2) - B(t,x_1',x_2')| \le K(|x_1 - x_1'| + |x_2 - x_2'|),$$
$$|A(t,x_1,x_2)| + |B(t,x_1,x_2)| \le L,$$

for any $x_1, x_2, x_1', x_2' \in \mathbb{R}^m$ and $t \ge 0$.

(A2) $A(t,x_1,x_2)$ is continuous with respect to (t,x_1,x_2).

(A3) There exists $c > 0$ such that

$$|v^T A(t,x_1,x_2)|^2 \ge c|v|^2,$$

for any $v \in \mathbb{R}^m, x_1, x_2 \in \mathbb{R}^m$, and $t \ge 0$.

Throughout this article, we use C or C_i, $i \in \mathbb{N}$ to denote a positive constant which may depend on constants K, L, d, m, p, and t.

First, let us prove a lemma on the existence of a unique solution to (1).

Lemma 1. *Assume* (A1). *Then* (1) *has a unique strong solution for any initial value $x \in \mathbb{R}^m$. Moreover, we have $E[|M_t^i|^p] \le C$ for any $t \ge 0$, $1 \le i \le m$, and $p \ge 2$.*

Proof. For $s \in [0,t]$, we use the Picard iteration method to define

$$X_s^{(0),i} := x^i$$
$$X_s^{(n+1),i} := x^i + \sum_{l=1}^d \int_0^s A_l^i(u, X_u^{(n)}, M_u^{(n)}) dW_u^l + \int_0^s B^i(u, X_u^{(n)}, M_u^{(n)}) du, \quad 1 \le i \le m, n \ge 0,$$

where $X_u^{(n)} := (X_u^{(n),1}, \dots, X_u^{(n),m})$ and $M_u^{(n)} := (\max_{v \le u} X_v^{(n),1}, \dots, \max_{v\le u} X_v^{(n),m})$. By (A1) it is straightforward to prove that $\{X_s^{(n)}, s \in [0,t]\}$ converges to the solution to (1) with $\lim_{n\to\infty} E[\max_{s \le t} |X_s^{(n)} - X_s|^2] = 0$ for any $t \ge 0$. The uniqueness follows from (A1) and Gronwall's lemma. Since $t \ge 0$ is arbitrary, we have the existence of a unique solution to (1). We have $E[|M_t^i|^p] \le C$ due to (A1). $\square$

Remark 1. Under (A1)–(A3), for any $t \geq 0$, $\{X_s^i, s \in [0, t]\}$ attains its maximum on $[0, t]$ on a unique point τ_t^i and $0 < \tau_t^i < t$, a.s. This follows along the lines of the ideas of the proofs of Propositions 1 and 18 in Hayashi and Kohatsu-Higa (2013).

Let us prove a lemma on the differentiability of the maximum of a continuous process which is similar to Proposition 2.1.10 of Nualart (2006).

Lemma 2. *For $t \geq 0$, let $\{\hat{X}_s, s \in [0, t]\}$ be a one-dimensional continuous process. Suppose that*

(i) $E[\sup_{s\leq t} |\hat{X}_s|^2] < \infty$,

(ii) *for any* $s \in [0, t]$, $\hat{X}_s \in \mathbb{D}^{1,2}$ *and* $E[\sup_{s\leq t} \|D\hat{X}_s\|_H^2] < \infty$.

Then $\hat{M}_t = \sup_{s\leq t} \hat{X}_s \in \mathbb{D}^{1,2}$, *and we have*

$$E\left[\|D\hat{M}_t\|_H^2\right] \leq E\left[\sup_{s\leq t} \|D\hat{X}_s\|_H^2\right]. \tag{2}$$

Moreover, if we assume that

(iii) $\{\hat{X}_s, s \in [0, t]\}$ *attains its maximum on a unique point* $\hat{\tau}_t^*$,

(iv) *for* $1 \leq j \leq d$, *and almost every* r, $\{D_r^j\hat{X}_s, s \in [0, t]\}$ *is continuous except for* $s = r$, *and*

(v) *for* $1 \leq j \leq d$, $E[\int_0^t \sup_{0\leq s\leq t} |D_r^j\hat{X}_s|^2 dr] < \infty$,

then we have

$$D_r\hat{M}_t = D_r\hat{X}_{\hat{\tau}_t^*}, \quad r\text{-a.e.}, \tag{3}$$

where we have defined $D_r\hat{X}_{\hat{\tau}_t^*} := D_r\hat{X}_s|_{s=\hat{\tau}_t^*}$.

Proof. Let $\{t_k\}_{k\geq 0}$ be a dense subset of $[0, t]$, and define

$$\hat{M}_t^n := \max\{\hat{X}_{t_1}, \ldots, \hat{X}_{t_n}\}.$$

Define

$$A_1 := \{\hat{X}_{t_1} = \hat{M}_t^n\}, \qquad A_k := \{\hat{X}_{t_1} \neq \hat{M}_t^n, \ldots, \hat{X}_{t_{k-1}} \neq \hat{M}_t^n, \hat{X}_{t_k} = \hat{M}_t^n\}, \quad 2 \leq k \leq n.$$

Then, by the local property of operator D, we have

$$D\hat{M}_t^n = \sum_{k=1}^n \mathbf{1}_{A_k} D\hat{X}_{t_k}.$$

By Proposition 2.1.10 of Nualart (2006), $\hat{M}_t = \sup_{0\leq s\leq t} \hat{X}_s$ belongs to $\mathbb{D}^{1,2}$ and $D\hat{M}_t^n \to D\hat{M}_t (n \to \infty)$ in the weak topology of $L^2(\Omega; L^2([0, t]; \mathbb{R}^d))$ under (i) and (ii). We obtain (2) from $E[\|D\hat{M}_t\|_H^2] \leq \liminf_{n\to\infty} E[\|D\hat{M}_t^n\|_H^2]$.

Let us prove (3). For $\omega \in A_k$, we define $\hat{\tau}_n^* := t_k$. Then $\hat{\tau}_n^* \to \hat{\tau}_t^*$, a.s., and we have

$$D\hat{M}_t^n = \sum_{k=1}^n \mathbf{1}_{A_k} D\hat{X}_{\hat{\tau}_n^*} = D\hat{X}_{\hat{\tau}_n^*},$$

where we have defined $D\hat{X}_{\hat{\tau}_n^*} := D\hat{X}_s|_{s=\hat{\tau}_n^*}$. Note that, if $r = \hat{\tau}_n^*$, then $D_r\hat{X}_{\hat{\tau}_n^*}$ is not well defined, due to the discontinuity; thus the rigorous meaning of the above equality is that $D_r\hat{M}_t^n = D_r\hat{X}_{\hat{\tau}_n^*}$, r-a.e. with probability 1.

Now let us prove that

$$E\left[\int_0^t \sum_{j=1}^d D_r^j\hat{X}_{\hat{\tau}_n^*} u_r^j dr\right] \to E\left[\int_0^t \sum_{j=1}^d D_r^j\hat{X}_{\hat{\tau}_t^*} u_r^j dr\right], \tag{4}$$

for any $u \in L^2(\Omega; L^2([0, t]; \mathbb{R}^d))$. We have

$$E\left[\int_0^t \sum_{j=1}^d D_r^j\hat{X}_{\hat{\tau}_n^*} u_r^j dr\right] - E\left[\int_0^t \sum_{j=1}^d D_r^j\hat{X}_{\hat{\tau}_t^*} u_r^j dr\right] = E\left[\int_0^t \sum_{j=1}^d (D_r^j\hat{X}_{\hat{\tau}_n^*} - D_r^j\hat{X}_{\hat{\tau}_t^*}) u_r^j dr\right]. \tag{5}$$

From (iv), we have $D_r^j\hat{X}_{\hat{\tau}_n^*} \to D_r^j\hat{X}_{\hat{\tau}_t^*}$ for $r \neq \hat{\tau}_t^*$ then $D_r^j\hat{X}_{\hat{\tau}_n^*} \to D_r^j\hat{X}_{\hat{\tau}_t^*}$, r-a.e. with probability 1. As $|D_r^j\hat{X}_{\hat{\tau}_n^*} - D_r^j\hat{X}_{\hat{\tau}_t^*}|^2 \leq 2\sup_{0\leq s\leq t} |D_r^j\hat{X}_s|^2$ and (v), we have

$$\int_0^t \sum_{j=1}^d |D_r^j\hat{X}_{\hat{\tau}_n^*} - D_r^j\hat{X}_{\hat{\tau}_t^*}|^2 dr \to 0, \text{ a.s.}$$

Due to $E[|D_r^j\hat{X}_{\hat{\tau}_n^*} - D_r^j\hat{X}_{\hat{\tau}_t^*}|^2] \leq 2E[\sup_{0\leq s\leq t}|D_r^j\hat{X}_s|^2]$ and (v), we have

$$\lim_{n\to\infty} E\left[\int_0^t \sum_{j=1}^d |D_r^j\hat{X}_{\hat{\tau}_n^*} - D_r^j\hat{X}_{\hat{\tau}_t^*}|^2\right] dr = 0.$$

Then we obtain (4). Since $D\hat{M}_t^n$ converges to $D\hat{M}_t$ weakly in $L^2(\Omega; L^2([0,t];\mathbb{R}^d))$ and (4) holds, we have

$$E\left[\int_0^t \sum_{j=1}^d \left(D_r^j\hat{X}_{\hat{\tau}_t^*} - D_r^j\hat{M}_t\right) u_r^j dr\right] = 0,$$

for any $u \in L^2(\Omega; L^2([0,t];\mathbb{R}^d))$. By the fact that $\hat{M}_t$ belongs to $\mathbb{D}^{1,2}$ and (v), we have $\{D_r\hat{X}_{\hat{\tau}_t^*} - D_r\hat{M}_t, r \in [0,t]\} \in L^2(\Omega; L^2([0,t];\mathbb{R}^d))$. Therefore we have (3) by taking $u_r = D_r\hat{X}_{\hat{\tau}_t^*} - D_r\hat{M}_t$, and this finishes the proof. □

Remark 2. In Lemma 2, if we assume that $\{\hat{X}_s, s\in[0,t]\}$ is adapted, then we have $D_r\hat{X}_{\hat{\tau}_t^*} = 0$ for almost every r such that $r > \hat{\tau}_t^*$. Thus, in this case, we can write $D_r\hat{M}_t = \mathbf{1}_{[0,\hat{\tau}_t^*)}(r)D_r\hat{X}_{\hat{\tau}_t^*}$, r-a.e.

Next, let us prove the differentiability of the solution to (1) in Malliavin sense.

Lemma 3. *Assume* (A1)–(A3). *Then, for $s\in[0,t]$ and $1\leq i\leq m$, X_s^i, M_s^i belong to $\mathbb{D}^{1,2}$. Moreover, $D_r^jX_s^i$ satisfies the following equation:*

$$D_r^jX_s^i = A_j^i(r, X_r, M_r) + \int_r^s \left[\bar{A}_{k,l}^i(u)D_r^jX_u^k + \tilde{A}_{k,l}^i(u)D_r^jM_u^k\right] dW_u^l + \int_r^s \left[\bar{B}_k^i(u)D_r^jX_u^k + \tilde{B}_k^i(u)D_r^jM_u^k\right] du, \tag{6}$$

for $r\leq s$, a.e., and

$$D_r^jX_s^i = 0, \tag{7}$$

for $r > s$, a.e., where $\bar{A}_{k,l}(u)$, $\tilde{A}_{k,l}(u)$, $\bar{B}_k(u)$, and $\tilde{B}_k(u)$ are uniformly bounded and adapted m-dimensional processes.

Proof. We will use the Picard approximation from Lemma 1, so $X_s^{(n)}$, $M_s^{(n)}$ are the processes constructed by recurrence there. The proof of this lemma uses the proof of Theorem 2.2.1 of Nualart (2006). We need to extend the proof to an equation with coefficients which depend on the maximum process. We start by proving that $X_s^{(n),i} \in \mathbb{D}^{1,2}$ for $s\in[0,t]$, $1\leq i\leq m$, and $n\geq 0$. If we assume that $X_s^{(n),i}\in\mathbb{D}^{1,2}$ and $E[\int_0^s \sup_{u\leq v}\|DX_u^{(n),i}\|_H^2 dv] < \infty$ for $s\in[0,t]$, then we have $M_s^{(n),i}\in\mathbb{D}^{1,2}$ for $s\in[0,t]$, and

$$E\left[\int_0^t\int_0^t |D_r^j(A_l^i(u, X_u^{(n)}, M_u^{(n)}))|^2 drdu\right] \leq C\sum_{k=1}^m E\left[\int_0^t \sup_{u\leq s}\|DX_u^{(n),k}\|_H^2 ds\right]$$

by (2). Therefore we have $X_s^{(n+1),i}\in\mathbb{D}^{1,2}$ for $s\in[0,t]$, and

$$D_r^jX_s^{(n+1),i} = A_j^i(r, X_r^{(n)}, M_r^{(n)}) + \int_r^s \left[\bar{A}_{k,l}^{(n),i}(u)D_r^jX_u^{(n),k} + \tilde{A}_{k,l}^{(n),i}(u)D_r^jM_u^{(n),k}\right] dW_u^l$$
$$+ \int_r^s \left[\bar{B}_k^{(n),i}(u)D_r^jX_u^{(n),k} + \tilde{B}_k^{(n),i}(u)D_r^jM_u^{(n),k}\right] du,$$

where $\bar{A}_{k,l}^{(n)}(u)$, $\tilde{A}_{k,l}^{(n)}(u)$, $\bar{B}_k^{(n)}(u)$ and $\tilde{B}_k^{(n)}(u)$ are uniformly bounded and adapted m-dimensional processes. From this expression, we have

$$\sum_{i=1}^m E\left[\sup_{u\leq s}\|DX_u^{(n+1),i}\|_H^2\right] \leq C_1 + C_2\int_0^s \sum_{i=1}^m E\left[\sup_{u\leq v}\|DX_u^{(n),i}\|_H^2\right] dv, \tag{8}$$

and this implies that $M_s^{(n+1),i}\in\mathbb{D}^{1,2}$ and $E[\int_0^s \sup_{u\leq v}\|DX_u^{(n+1),i}\|_H^2 dv] < \infty$ for $s\in[0,t]$, by Lemma 2.

Due to (8) and Lemma 2, we have $\sup_n E[\|DX_s^{(n),i}\|_H^2] < \infty$ and $\sup_n E[\|DM_s^{(n),i}\|_H^2] \leq \sup_n E[\sup_{u\leq s}\|DX_u^{(n),i}\|] < \infty$. By the fact that $X_s^{(n),i}\to X_s^i$, $M_s^{(n),i}\to M_s^i$ in $L^2(\Omega)$ and Lemma 1.2.3 of Nualart (2006), X_s^i and M_s^i belong to $\mathbb{D}^{1,2}$ for $s\in[0,t]$. Moreover, $DX_s^{(n),i}$ and $DM_s^{(n),i}$ converge to DX_s^i and DM_s^i in the weak topology of $L^2(\Omega; L^2([0,t];\mathbb{R}^d))$. Let us prove (6). We have

$$E\left[\int_0^t\int_0^t |D_r^j(A_l^i(u, X_u, M_u))|^2 drdu\right] \leq C\sum_{k=1}^m \int_0^t E\left[\|DX_u^k\|_H^2 + \|DM_u^k\|_H^2\right] du < \infty$$

by $E[\|DM_u^k\|_H^2] \leq \liminf_{n\to\infty} E[\|DM_u^{(n),k}\|_H^2] \leq \liminf_{n\to\infty} E[\sup_{v\leq u}\|DX_v^{(n),k}\|_H^2]$ and (8). Therefore we have (6), and the proof is completed. □

Lemma 4. *Assume* (A1)–A3) *. Then, for* $\{X_s^i, s\in[0,t]\}$ *and* $p\geq 2$*, we have*

$$E\left[\int_0^s \sup_{r\leq u\leq s}|D_r^j X_u^i|^p dr\right] < \infty, \tag{9}$$

and assumptions (i)–(v) *of* Lemma 2 *hold. Moreover, for* $s\in[0,t]$ *and* $p\geq 2$, $X_s^i, M_s^i \in \mathbb{D}^{1,p}$.

Proof. First, let us prove (9) for $p=2$. We have

$$\sum_{j=1}^d E\left[\int_0^s \sup_{r\leq u\leq s}|D_r^j X_u^i|^2 dr\right] \leq C_1 + C_2\sum_{k=1}^m \int_0^s E\left[\|DX_v^k\|_H^2 + \|DM_v^k\|_H^2\right]dv < \infty,$$

by (6) and (8) and $E[\|DM_v^k\|_H^2] \leq \liminf_{n\to\infty} E[\|DM_v^{(n),k}\|_H^2] \leq \liminf_{n\to\infty} E[\sup_{u\leq v}\|DX_u^{(n),k}\|_H^2]$. This implies that (v) holds for X^i. (i) follows from Lemma 1, and we have (ii) by (9) for $p=2$. (iii) holds due to Remark 1, and we have (iv) by (6) and (7).

Let us prove (9) for $p>2$. It suffices to prove that

$$E\left[\int_0^t \sum_{i=1}^m\sum_{j=1}^d \sup_{r\leq s\leq t}|D_r^j X_s^i|^p dr\right] \leq C_1 + C_2\int_0^t E\left[\int_0^u \sum_{i=1}^m\sum_{j=1}^d \sup_{r\leq s\leq u}|D_r^j X_s^i|^p dr\right]du, \tag{10}$$

but we have (10) easily by (3), (6) and (7). From (9), we have $X_s^i, M_s^i\in\mathbb{D}^{1,p}$ for $s\in[0,t]$ and $p\geq 2$. □

Now we consider two $m\times m$ matrix-valued processes defined by

$$Y_j^i(s) = \delta_j^i + \int_0^s \bar{A}_{k,l}^i(u)Y_j^k(u)dW_u^l + \int_0^s \bar{B}_k^i(u)Y_j^k(u)du, \quad 1\leq i,j\leq m \tag{11}$$

and

$$Z_j^i(s) = \delta_j^i - \int_0^s Z_k^i(u)\bar{A}_{j,l}^k(u)dW_u^l - \int_0^s Z_k^i(u)\left[\bar{B}_j^k(u) - \bar{A}_{\alpha,l}^k(u)\bar{A}_{j,l}^\alpha(u)\right]du, \quad 1\leq i,j\leq m. \tag{12}$$

By the argument in Section 2.3 of Nualart (2006), we have $Y^{-1}(s) = Z(s)$. Let us express $D_r^j X_s^i$ in terms of $Y(s)$ and $Z(s)$.

Lemma 5. *For* $s\in[r,t]$ *and* $1\leq i\leq m$, $1\leq j\leq d$, $D_r^j X_s^i$ *satisfies*

$$D_r^j X_s^i = Y_k^i(s)Z_{k'}^k(r)A_j^{k'}(r) + Y_k^i(s)\int_r^s Z_{k'}^k(u)\tilde{A}_{l',l}^{k'}(u)D_r^j M_u^{l'}dW_u^l + Y_k^i(s)\int_r^s Z_{k'}^k(u)\left[\tilde{B}_{l'}^{k'}(u) - \bar{A}_{\alpha,l}^{k'}\tilde{A}_{l',l}^{\alpha}(u)\right]D_r^j M_u^{l'}du. \tag{13}$$

Proof. We have (13) easily by (6), (11), (12), and Itô's formula. □

Now we prove the absolute continuity of the law of X_t, which is the main theorem of this section.

Theorem 1. *Assume* (A1)–(A3)*. Then, for* $t>0$*, the law of* X_t *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}^m$*.*

Proof. Let us prove $\int_0^t |v^T D_r X_t|^2 dr > 0$ for nonzero vector $v\in\mathbb{R}^m$. By (13), we have

$$\begin{aligned}|v^T D_r X_t|^2 \geq\ & \frac{1}{2}\sum_{j=1}^d\left|\sum_{i=1}^m v_i Y_k^i(t)Z_{k'}^k(r)A_j^{k'}(r)\right|^2 \\ & - \sum_{j=1}^d\left|\sum_{i=1}^m v_i Y_k^i(t)\left(\int_r^t Z_{k'}^k(s)\tilde{A}_{l',l}^{k'}(s)D_r^j M_s^{l'}dW_s^l + \int_r^t Z_{k'}^k(s)[\tilde{B}_{l'}^{k'}(s) - \bar{A}_{\alpha,l}^{k'}\tilde{A}_{l',l}^{\alpha}(s)]D_r^j M_s^{l'}ds\right)\right|^2 \\ =:\ & \frac{1}{2}\sum_{j=1}^d\left|\sum_{i=1}^m v_i Y_k^i(t)Z_{k'}^k(r)A_j^{k'}(r)\right|^2 + \mathcal{A}_{r,t}.\end{aligned} \tag{14}$$

By Schwartz's inequality, Burkholder–Davis–Gundy's inequality, and Lemma 4, we have

$$E\left[\int_{t-\varepsilon}^{t}\left|Y_k^i(t)\int_r^t Z_{k'}^k(s)[\tilde{B}_{l'}^{k'}(s)-\bar{A}_{\alpha,l}^{k'}(s)\tilde{A}_{l',l}^{\alpha}(s)]D_r^jM_s^{l'}ds\right|^2dr\right]$$
$$\leq \varepsilon^{\frac{3}{2}}CE\left[|Y(t)|^8\right]^{\frac{1}{4}}E\left[\sup_{s\leq t}|Z(s)|^8\right]^{\frac{1}{4}}\sum_{l'=1}^{m}\left(E\left[\int_0^t\sup_{r\leq s\leq t}|D_r^jX_s^{l'}|^4dr\right]\right)^{\frac{1}{2}}<\infty$$

and

$$E\left[\int_{t-\varepsilon}^{t}\left|Y_k^i(t)\int_r^t Z_{k'}^k(s)\tilde{A}_{l',l}^{k'}(s)D_r^jM_s^{l'}dW_s^l\right|^2\right]$$
$$\leq \varepsilon^{\frac{3}{2}}CE\left[|Y(t)|^4\right]^{\frac{1}{2}}E\left[\sup_{s\leq t}|Z(s)|^8\right]^{\frac{1}{4}}\sum_{l'=1}^{m}\left(E\left[\int_0^t\sup_{r\leq u\leq t}|D_r^jX_s^{l'}|^8dr\right]\right)^{\frac{1}{4}}<\infty.$$

This shows that $\frac{1}{\varepsilon}\int_{t-\varepsilon}^t \mathcal{A}_{r,t}dr\to 0$ in $L^1(\Omega)$ as ε tends to 0. Therefore, there exists $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that $\varepsilon_n\searrow 0(n\to\infty)$ and

$$\lim_{n\to\infty}\frac{1}{\varepsilon_n}\int_{t-\varepsilon_n}^t\mathcal{A}_{r,t}dr=0,\text{ a.s.}$$

On the other hand, by the continuity of A_j^i, we have

$$\lim_{n\to\infty}\frac{1}{\varepsilon_n}\int_{t-\varepsilon_n}^t\sum_{j=1}^d\left|\sum_{i=1}^m v_iY_k^i(t)Z_{k'}^k(r)A_j^{k'}(r)\right|^2=\sum_{j=1}^d\left|\sum_{i=1}^m v_iA_j^i(t,X_t,M_t)\right|^2>0,$$

for any nonzero vector $v\in\mathbb{R}^m$ by (A3). By Lemma 4 and Theorem 2.1.2 of Nualart (2006), the proof is completed. □

3. The absolute continuity of the probability law of $(X_t^i, M_t^{i'})$

In this section, we prove the absolute continuity of the law of $(X_t^i, M_t^{i'})$, $1\leq i, i'\leq m$, in a special case. That is:

(A4) A_l^i, $1\leq i\leq m$, $1\leq l\leq d$, do not depend on the second space variable,

in addition to (A1)–(A3).

Remark 3. Under (A1)–(A4), $\tilde{A}_{l',l}^{k'}(u)\equiv 0$ in (13).

Theorem 2. *Assume* (A1)–(A4)*. Then, for $t>0$ and $1\leq i, i'\leq m$, the law of $(X_t^i, M_t^{i'})$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2$.*

Proof. Let $v_1, v_2\in\mathbb{R}\setminus\{0\}$. Note that, by Lemmas 2 and 4, for $t>0$, we have $D_r^jM_t^{i'}=\mathbf{1}_{[0,\tau_t^{i'})}(r)D_r^jX_s^{i'}|_{s=\tau_t^{i'}}$. For simplicity of notation, we define $D_r^jX_{\tau_t^{i'}}^{i'}:=D_r^jX_s^{i'}|_{s=\tau_t^{i'}}$. First, we assume that $v_1\neq 0$, $v_2\neq 0$. By Schwartz's inequality and the trivial inequality $a^2+b^2\geq 2ab$, $a,b\in\mathbb{R}$, we have

$$\int_0^t\left|(v_1,v_2)\begin{pmatrix}D_r^1X_t^i\cdots D_r^dX_t^i\\ D_r^1M_t^{i'}\cdots D_r^dM_t^{i'}\end{pmatrix}\right|^2dr$$
$$=\int_0^t\sum_{j=1}^d|v_1D_r^jX_t^i|^2dr+2\int_0^{\tau_t^{i'}}\sum_{j=1}^d v_1D_r^jX_t^iv_2D_r^jX_{\tau_t^{i'}}^{i'}dr+\int_0^{\tau_t^{i'}}\sum_{j=1}^d|v_2D_r^jX_{\tau_t^{i'}}^{i'}|^2dr$$
$$\geq\int_0^t\sum_{j=1}^d|v_1D_r^jX_t^i|^2dr-2\left(\int_0^{\tau_t^{i'}}\sum_{j=1}^d|v_1D_r^jX_t^i|^2dr\right)^{\frac{1}{2}}\left(\int_0^{\tau_t^{i'}}\sum_{j=1}^d|v_2D_r^jX_{\tau_t^{i'}}^{i'}|^2dr\right)^{\frac{1}{2}}+\int_0^{\tau_t^{i'}}\sum_{j=1}^d|v_2D_r^jX_{\tau_t^{i'}}^{i'}|^2dr$$
$$\geq 2\left(\int_0^t\sum_{j=1}^d|v_1D_r^jX_t^i|^2dr\right)^{\frac{1}{2}}\left(\int_0^{\tau_t^{i'}}\sum_{j=1}^d|v_2D_r^jX_{\tau_t^{i'}}^{i'}|^2dr\right)^{\frac{1}{2}}$$
$$-2\left(\int_0^{\tau_t^{i'}}\sum_{j=1}^d|v_1D_r^jX_t^i|^2dr\right)^{\frac{1}{2}}\left(\int_0^{\tau_t^{i'}}\sum_{j=1}^d|v_2D_r^jX_{\tau_t^{i'}}^{i'}|^2dr\right)^{\frac{1}{2}}$$

$$= 2|v_1 \parallel v_2| \left(\int_0^{\tau_t^{i'}} \sum_{j=1}^d |D_r^j X_{\tau_t^{i'}}^{i'}|^2 dr \right)^{\frac{1}{2}} \left[\left(\int_0^t \sum_{j=1}^d |D_r^j X_t^i|^2 dr \right)^{\frac{1}{2}} - \left(\int_0^{\tau_t^{i'}} \sum_{j=1}^d |D_r^j X_t^i|^2 dr \right)^{\frac{1}{2}} \right]. \quad (15)$$

By (A4) and the same calculation as in the proof of Theorem 1, we can prove that there exist $\{\varepsilon_n^i\}_{n\in\mathbb{N}}$ and $\{\varepsilon_n^{i'}\}_{n\in\mathbb{N}}$ such that

$$\lim_{\varepsilon_n^i \searrow 0} \frac{1}{\varepsilon_n^i} \int_{t-\varepsilon_n^i}^t \sum_{j=1}^d |D_r^j X_t^i|^2 dr \geq \frac{1}{2} \sum_{j=1}^d |A_j^i(t, X_t)|^2 \quad (16)$$

and

$$\lim_{\varepsilon_n^{i'} \searrow 0} \frac{1}{\varepsilon_n^{i'}} \int_{\tau_t^{i'}-\varepsilon_n^{i'}}^{\tau_t^{i'}} \sum_{j=1}^d |D_r^j X_{\tau_t^{i'}}^{i'}|^2 dr \mathbf{1}_{\{\tau_t^{i'}-\varepsilon_n^{i'}>0\}} \geq \frac{1}{2} \sum_{j=1}^d |A_j^{i'}(\tau_t^{i'}, X_{\tau_t^{i'}})|^2, \quad (17)$$

almost surely. As $t > \tau_t^i$, there exists $N' \in \mathbb{N}$ such that

$$\int_{t-\varepsilon_n^i}^t \sum_{j=1}^d |D_r^j X_t^i|^2 dr \geq \frac{\varepsilon_n^i}{2} \sum_{j=1}^d |A_j^i(t, X_t)|^2,$$

$$\int_{\tau_t^{i'}-\varepsilon_t^{i'}}^{\tau_t^{i'}} \sum_{j=1}^d |D_r^j X_{\tau_t^{i'}}^{i'}|^2 dr \geq \frac{\varepsilon_n^{i'}}{2} \sum_{j=1}^d |A_j^{i'}(\tau_t^{i'}, X_{\tau_t^{i'}})|^2,$$

$$t - \varepsilon_n^i > \tau_t^{i'},$$

for any $n \geq N'$, almost surely. This implies that the right-hand side of (15) is strictly positive for any $v \in \mathbb{R}^2$ such that $v_1 \neq 0$, $v_2 \neq 0$. Second, when $v_1 = 0$ or $v_2 = 0$, we have

$$\int_0^t \left| (v_1, v_2) \begin{pmatrix} D_r^1 X_t^i \cdots D_r^d X_t^i \\ D_r^1 M_t^{i'} \cdots D_r^d M_t^{i'} \end{pmatrix} \right|^2 dr > 0, \text{ a.s.}$$

by (16) and (17). This finishes the proof. □

Remark 4. The general $2m$-dimensional study of the law of $(X_t^1, \ldots, X_t^m, M_t^1, \ldots, M_t^m)$ does not follow with the arguments presented here, due to the particular structure used in the calculation of (15).

Corollary 1. *Under* (A1)–(A4), *by the same calculation as that in Theorem 2, for $t > 0$ and $1 \leq i \neq i' \leq m$, we can prove the absolute continuity of the law of $(M_t^i, M_t^{i'})$ conditioned by the set $\{\tau_t^i \neq \tau_t^{i'}\}$.*

Now, we give an example for A_l^i and $A_l^{i'}$ that $\{\tau_t^i \neq \tau_t^{i'}\}$ holds, a.s.

Example 1. For each $k = i, i'$, let $\{X_s^k, s \in [0, t]\}$ satisfy

$$X_s^k = x^k + \int_0^s B^k(u, X_u, M_u) du + A_k^k W_s^k,$$

where A_k^k is a nonzero constant. Then $\tau_t^i \neq \tau_t^{i'}$, a.s.

Proof. By Girsanov's theorem, the independence of Brownian motions, and the explicit density function for τ_t^k, $k = i, i'$ (Problem 8.17 in Chapter 2 of Karatzas and Shreve (1991)), we obtain the existence of the density function for $\tau_t^i - \tau_t^{i'}$. Then we have $P(\tau_t^i = \tau_t^{i'}) = 0$. □

4. A concluding remark

In this article, we have proved the absolute continuity of the laws of X_t and $(X_t^i, M_t^{i'})$ with Lipschitz continuous coefficients under some additional assumptions. We end this article with some remarks on the law of the maximum of processes. There are some theoretical and applicable results about the law of the maximum of continuous processes. In Nualart (2006), the smoothness of the density function of the maximum of a Wiener sheet is proved. In Gobet and Kohatsu-Higa (2003), the authors derived some integration by parts formulas involving the maximum and minimum of a one-dimensional diffusion to compute the sensitivities of the price of financial products with respect to market parameters called Greeks. Recently, the smoothness of the density function of the joint law of a multi-dimensional diffusion at the time when a component attains its maximum was proved in Hayashi and Kohatsu-Higa (2013). In these articles, Garsia–Rodemich–Rumsey's lemma (Lemma A.3.1 of Nualart (2006)) plays an important role in obtaining the results. A more recent article, Yue and Zhang (in press), showed that the laws of a perturbed SDE and a perturbed reflected SDE are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.

Acknowledgments

The author would like to thank Arturo Kohatsu-Higa for helpful comments. Also, the author is particularly grateful to the reviewers for constructive comments and suggestions on the article.

References

Fournié, E., Lasry, J.M., Lebuchoux, J., Lions, P.L., Touzi, N., 1999. Applications of Malliavin calculus to Monte Carlo methods in finance. Finance and Stochastic 3, 391–412.
Fournier, N., Printems, J., 2010. Absolute continuity for some one-dimensional processes. Bernoulli 16 (2), 343–360.
Gobet, E., Kohatsu-Higa, A., 2003. Computation of Greeks for barrier and look-back options using Malliavin calculus. Electronic Communications in Probability 8, 51–62.
Hayashi, M., Kohatsu-Higa, A., 2013. Smoothness of the distribution of the supremum of a multi-dimensional diffusion process. Potential Analysis 38/1, 57–77.
Karatzas, I., Shreve, S., 1991. Brownian Motion and Stochastic Calculus, second ed. In: Graduate Texts in Mathematics, vol. 113. Springer-Verlag, New York.
Nualart, D., 2006. The Malliavin Calculus and Related Topics, second ed. Springer-Verlag, Berlin.
Yue, W., Zhang, T.S., 2013. Absolute continuity of the laws of perturbed diffusion processes and perturbed reflected diffusion processes. Journal of Theoretical Probability http://dx.doi.org/10.1007/s10959-013-0499-7 (in press).