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A Tauberian theorem for Cesàro summability of integrals

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ABSTRACT

In this paper we give a proof of the generalized Littlewood Tauberian theorem for Cesàro summability of improper integrals.

1. Introduction

Throughout this paper we assume that f is a real-valued function which is continuous on $[0, \infty)$. Let $s(x) = \int_0^x f(t)\mathrm{d}t$. The Cesàro means of $s(x)$ are defined by

$$\sigma(s(x)) = \frac{1}{x}\int_0^x s(t)\mathrm{d}t.$$

The integral

$$\int_0^\infty f(t)\mathrm{d}t$$

is said to be Cesàro summable to a finite number L if

$$\lim_{x\to\infty} \sigma(s(x)) = \lim_{x\to\infty}\int_0^x \left(1-\frac{t}{x}\right) f(t)\mathrm{d}t = L. \tag{1}$$

If the integral

$$\int_0^\infty f(t)\mathrm{d}t = L \tag{2}$$

exists, then limit (1) also exists. The converse is not necessarily true. Adding some suitable condition to (1) which is called a Tauberian condition may imply (2). Any theorem which states that the convergence of the integral follows from the Cesàro summability of the integral and some Tauberian condition is said to be a Tauberian theorem.

For a function $s(x) = \int_0^x f(t)\mathrm{d}t$, we have

$$s(x) - \sigma(s(x)) = v(x) \tag{3}$$

where $v(f(x)) = \frac{1}{x}\int_0^x tf(t)\mathrm{d}t$. Note that $\sigma'(s(x)) = \frac{v(f(x))}{x}$.

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Define $\sigma_k(s(x))$ for each nonnegative integer k by

$$\sigma_k(s(x)) = \begin{cases} \dfrac{1}{x}\displaystyle\int_0^x \sigma_{k-1}(s(t))\mathrm{d}t, & k \geq 1 \\ s(x), & k = 0. \end{cases}$$

Note that $\sigma_1(s(x)) = \sigma(s(x))$.

De la Vallée Poussin means of $\int_0^x f(t)\mathrm{d}t$ are defined by

$$\tau(s(x)) = \frac{1}{\lambda x - x}\int_x^{\lambda x} s(t)\mathrm{d}t$$

for $\lambda > 1$, and

$$\tau(s(x)) = \frac{1}{x - \lambda x}\int_{\lambda x}^{x} s(t)\mathrm{d}t$$

for $0 < \lambda < 1$.

A real-valued function $s(x) = \int_0^x f(t)\mathrm{d}t$ is slowly oscillating in the sense of Stanojević [1] if

$$\lim_{\lambda\to 1^+}\limsup_{x\to\infty}\max_{x\leq t\leq\lambda x}|s(t) - s(x)| = 0. \tag{4}$$

An equivalent reformulation of (4) can be given as follows:

$$\lim_{\lambda\to 1^-}\limsup_{x\to\infty}\max_{\lambda x\leq t\leq x}|s(t) - s(x)| = 0. \tag{5}$$

We note that for sequences of real numbers, an analogous definition was introduced by Stanojević [1]. Using this definition, Çanak [2] gave an alternative proof of generalized Littlewood Tauberian theorem for Abel summable sequences.

The aim of this paper is to prove the following generalized Littlewood Tauberian theorem for Cesàro summability of improper integrals:

Theorem 1. *If $s(x)$ is Cesàro summable to s and $s(x)$ is slowly oscillating, then* $\lim_{x\to\infty} s(x) = s$.

2. Lemmas

We need the following lemmas to prove our main theorem.

An equivalent definition of slow oscillation of $s(x)$ is given in terms of $v(x)$ by the following lemma.

Lemma 2. *$s(x)$ is slowly oscillating if and only if $v(x)$ is slowly oscillating and bounded.*

Proof. Suppose that $s(x)$ is slowly oscillating. We first show that $v(f(x)) = O(1)$, $x \to \infty$. It is clear that

$$\int_0^x uf(u)\mathrm{d}u = \sum_{j=0}^{\infty}\int_{x/2^{j+1}}^{x/2^j} uf(u)\mathrm{d}u. \tag{6}$$

It follows from the identity

$$\begin{aligned}\int_\alpha^\beta uf(u)\mathrm{d}u &= \int_\alpha^\beta us'(u)\mathrm{d}u = [us(u)]_\alpha^\beta - \int_\alpha^\beta s(u)\mathrm{d}u \\ &= -\int_\alpha^\beta s(u)\mathrm{d}u + \beta s(\beta) - \alpha s(\alpha) - \alpha s(\beta) + \alpha s(\beta) \\ &= -\int_\alpha^\beta s(u)\mathrm{d}u + (\beta-\alpha)s(\beta) + \alpha(s(\beta) - s(\alpha)) \\ &= -\int_\alpha^\beta (s(u) - s(\beta))\mathrm{d}u + \alpha(s(\beta) - s(\alpha))\end{aligned}$$

that

$$\begin{aligned}\left|\int_\alpha^\beta uf(u)\mathrm{d}u\right| &\leq (\beta-\alpha)\max_{\alpha\leq x\leq\beta}|s(x) - s(\beta)| + \alpha|s(\beta) - s(\alpha)| \\ &\leq \beta\max_{\alpha\leq x\leq\beta}|s(x) - s(\beta)|.\end{aligned}$$

If we choose $\beta = \frac{x}{2^j}$ and $\frac{\beta}{\alpha} \le 2$, we have

$$\left| \int_0^x uf(u)\mathrm{d}u \right| \le k \sum_{j=0}^{\infty} \frac{x}{2^j} = O(x), \quad x \to \infty. \tag{7}$$

We now show that $\sigma(s(x))$ is slowly oscillating. Since $\sigma'(s(x)) = \frac{v(f(x))}{x}$, we have

$$|\sigma(s(t)) - \sigma(s(x))| = \left| \int_x^t \sigma'(s(u))\mathrm{d}u \right| = \left| \int_x^t f(u)\mathrm{d}u \right| \le C \int_x^t \frac{\mathrm{d}u}{u} = C \log \frac{t}{x}$$

for any $x \le t \le \lambda x$, whence we conclude that $\max_{x \le t \le \lambda x} |\sigma(s(t)) - \sigma(s(x))| \le C \log \lambda$. Taking the limit of both sides as $\lambda \to 1^+$, we obtain

$$\lim_{\lambda \to 1^+} \limsup_{x \to \infty} \max_{x \le t \le \lambda x} |\sigma(s(t)) - \sigma(s(x))| = 0.$$

It follows by Kronecker identity (3) that $v(f(x))$ is slowly oscillating.

Conversely, suppose that $v(f(x))$ is bounded and slowly oscillating. It is clear that boundedness of $v(f(x))$ implies slow oscillation of $\sigma(s(x))$. Since $v(f(x))$ is slowly oscillating, it follows by Kronecker identity (3) that $s(x)$ is slowly oscillating. □

We represent the difference $s(x) - \sigma(s(x))$ in two different ways.

Lemma 3.

(i) *For* $\lambda > 1$,

$$s(x) - \sigma(s(\lambda x)) = \frac{1}{\lambda - 1}(\sigma(s(\lambda x)) - \sigma(s(x))) - \frac{1}{\lambda x - x} \int_x^{\lambda x} (s(t) - s(x))\mathrm{d}t.$$

(ii) *For* $0 < \lambda < 1$,

$$s(x) - \sigma(s(\lambda x)) = \frac{1}{1 - \lambda}(\sigma(s(x)) - \sigma(s(\lambda x))) + \frac{1}{x - \lambda x} \int_{\lambda x}^{x} (s(x) - s(t))\mathrm{d}t.$$

Proof. (i) From the definition of de la Vallée Poussin means of $s(x)$, we have

$$\tau(s(x)) = \frac{1}{\lambda x - x} \int_x^{\lambda x} s(t)\mathrm{d}t = \frac{1}{x(\lambda - 1)} \left(\int_0^{\lambda x} s(t)\mathrm{d}t - \int_0^x s(t)\mathrm{d}t \right)$$

for $\lambda > 1$. Since $\sigma(s(\lambda x)) = \frac{1}{\lambda x} \int_0^{\lambda x} s(t)\mathrm{d}t$, and $\sigma(s(x)) = \frac{1}{x} \int_0^x s(t)\mathrm{d}t$, we obtain

$$\tau(s(x)) = \frac{\lambda}{\lambda - 1}\sigma(s(\lambda x)) - \frac{1}{\lambda - 1}\sigma(s(x)) = \left(1 + \frac{1}{\lambda - 1}\right)\sigma(s(\lambda x)) - \frac{1}{\lambda - 1}\sigma(x).$$

The difference $\tau(s(x)) - \sigma(s(\lambda x))$ can be written as

$$\tau(s(x)) - \sigma(s(\lambda x)) = \frac{1}{\lambda - 1}\sigma(s(\lambda x)) - \frac{1}{\lambda - 1}\sigma(s(x)). \tag{8}$$

Subtracting $\sigma(s(\lambda x))$ from the identity $s(x) = \tau(x) - \frac{1}{\lambda x - x} \int_x^{\lambda x} (s(t) - s(x))\mathrm{d}t$, we get

$$s(x) - \sigma(s(\lambda x)) = (\tau(x) - \sigma(s(\lambda x))) - \frac{1}{\lambda x - x} \int_x^{\lambda x} (s(t) - s(x))\mathrm{d}t. \tag{9}$$

Using identity (8) we have

$$s(x) - \sigma(s(\lambda x)) = \frac{1}{\lambda - 1}(\sigma(s(\lambda x)) - \sigma(s(x))) - \frac{1}{\lambda x - x} \int_x^{\lambda x} (s(t) - s(x))\mathrm{d}t. \tag{10}$$

This completes the proof. □

(ii) Proof of Lemma 3(ii) is similar to that of Lemma 3(i).

3. Proof of Theorem 1

Proof. Since $s(x)$ is Cesàro summable to s, then $\sigma(s(x))$ is also Cesàro summable to s. Hence, it follows from (3) that $v(x)$ is Cesàro summable to zero. It follows by Lemma 2 that $v(x)$ is slowly oscillating. By Lemma 3(i) we have

$$v(x) - \sigma(v(\lambda x)) = \frac{1}{\lambda - 1}(\sigma(v(\lambda x)) - \sigma(v(x))) - \frac{1}{\lambda x - x}\int_x^{\lambda x}(v(t) - v(x))\mathrm{d}t. \tag{11}$$

By (11)

$$|v(x) - \sigma(v(x))| \le \frac{1}{\lambda - 1}|\sigma(v(\lambda x)) - \sigma(v(x))| + \max_{x \le t \le \lambda x}|v(t) - v(x)|. \tag{12}$$

Taking the lim sup of both sides of (12) as $x \to \infty$, we have

$$\limsup_{x\to\infty}|v(x) - \sigma(v(x))| \le \frac{1}{\lambda - 1}\limsup_{x\to\infty}|\sigma(v(\lambda x)) - \sigma(v(x))| + \limsup_{x\to\infty}\max_{x \le t \le \lambda x}|v(t) - v(x)|. \tag{13}$$

Since $\sigma(v(x))$ converges, the first term on the right-hand side of (13) vanishes and (13) becomes

$$\limsup_{x\to\infty}|v(x) - \sigma(v(x))| \le \limsup_{x\to\infty}\max_{x \le t \le \lambda x}|v(t) - v(x)|. \tag{14}$$

Letting $\lambda \to 1^+$ in (14), we have $\limsup_{x\to\infty}|v(x) - \sigma(v(x))| \le 0$. This implies that $v(x) = o(1)$ as $x \to \infty$. Since $s(x)$ is Cesàro summable to s and $v(x) = o(1)$ as $x \to \infty$, $\lim_{x\to\infty} s(x) = s$. This completes the proof. □

Note that Theorem 1 can be also proved similarly by using the equivalent reformulation (5) of (4) and Lemma 3(ii).

Corollary 4. *If $s(x)$ is Cesàro summable to s and $p(x)f(x) = O(p'(x))$, $x \to \infty$, where*

$$\lim_{\lambda\to 1^+}\limsup_{x\to\infty}\frac{p(\lambda x)}{p(x)} = 1, \tag{15}$$

then $\lim_{x\to\infty} s(x) = s$.

Proof. □

For any $x \le t \le \lambda x$, we have

$$|s(t) - s(x)| = \left|\int_x^t f(u)\mathrm{d}u\right| \le C\int_x^t \frac{p'(u)}{p(u)}\mathrm{d}u = C\log\frac{p(t)}{p(x)},$$

whence we conclude that $\limsup_{x\to\infty}\max_{x\le t\le\lambda x}|s(t) - s(x)| \le C\log\limsup_{x\to\infty}\frac{p(\lambda x)}{p(x)}$. Taking the limit of both sides as $\lambda \to 1^+$, we obtain

$$\lim_{\lambda\to 1^+}\limsup_{x\to\infty}\max_{x\le t\le\lambda x}|s(t) - s(x)| = 0$$

i.e. $s(x)$ is slowly oscillating.

Corollary 5 ([3]). *If $s(x)$ is Cesàro summable to s and $xf(x) = O(1)$, $x \to \infty$, then $\lim_{x\to\infty} s(x) = s$.*

Proof. Choose $p(x) = x$ in Corollary 4. □

Finally, we show that slow oscillation of $v(x)$ is also a Tauberian condition for Cesàro summability of improper integrals.

Theorem 6. *If $s(x)$ is Cesàro summable to s and $v(x)$ is slowly oscillating, then $\lim_{x\to\infty} s(x) = s$.*

Proof. Since $s(x)$ is Cesàro summable to s, then $\sigma(s(x))$ is also Cesàro summable to s. Hence, it follows from (3) that $v(x)$ is Cesàro summable to zero. Applying identity (3) to $v(x)$, we have $v(v(x))$ is Cesàro summable to zero. By Lemma 3(i) we have

$$v(v(x)) - \sigma(v(v(\lambda x))) = \frac{1}{\lambda - 1}(\sigma(v(v(\lambda x))) - \sigma(v(v(x)))) - \frac{1}{\lambda x - x}\int_x^{\lambda x}(v(v(t)) - v(v(x)))\mathrm{d}t. \tag{16}$$

By (16)

$$|v(v(x)) - \sigma(v(v(x)))| \le \frac{1}{\lambda - 1}|\sigma(v(v(\lambda x))) - \sigma(v(v(x)))| + \max_{x\le t\le\lambda x}|v(v(t)) - v(v(x))|. \tag{17}$$

Taking the lim sup of both sides of (17) as $x \to \infty$, we have

$$\limsup_{x\to\infty} |v(v(x)) - \sigma(v(v(x)))| \le \frac{1}{\lambda - 1} \limsup_{x\to\infty} |\sigma(v(v(\lambda x))) - \sigma(v(v(x)))| + \limsup_{x\to\infty} \max_{x\le t\le \lambda x} |v(v(t)) - v(v(x))|. \quad (18)$$

Since $\sigma(v(v(x)))$ converges, the first term on the right-hand side of (18) vanishes and (18) becomes

$$\limsup_{x\to\infty} |v(v(x)) - \sigma(v(v(x)))| \le \limsup_{x\to\infty} \max_{x\le t\le \lambda x} |v(v(t)) - v(v(x))|. \quad (19)$$

Letting $\lambda \to 1^+$ in (19), we have $\limsup_{x\to\infty} |v(v(x)) - \sigma(v(v(x)))| \le 0$. This implies that $v(v(x)) = o(1)$ as $x \to \infty$. From identity $v(x) - \sigma(v(x)) = v(v(x))$, we obtain $v(x) = o(1)$. Since $s(x)$ is Cesàro summable to s and $v(x) = o(1)$ as $x \to \infty$, $\lim_{x\to\infty} s(x) = s$. This completes the proof. □

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