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Superfidelity and trace distance

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ABSTRACT

Trace distance and superfidelity play an important role in quantum information theory. The aim of this Letter is to consider an inequality involving trace distance and superfidelity in infinite dimension and give a necessary and sufficient condition for equality of this inequality. In addition, some related results involving trace distance and superfidelity are obtained.

1. Introduction

In quantum information theory, an essential task is to distinguish two quantum states. One of the important tools is the trace metric, another tool is quantum fidelity [1–9]. Both are widely used by the quantum information community and have found applications in a number of problems such as quantum cryptography [10] and quantum phase transitions [11]. The trace distance is also related to the von Neumann entropy and relative entropy [3].

The mathematical description of a quantum mechanical system is based on a complex Hilbert space $\mathcal{H}$, with bounded quantum observables being represented by bounded self-adjoint operators A acting on $\mathcal{H}$. Let $\mathcal{T}(\mathcal{H})$ be the set of all trace class operators on $\mathcal{H}$, $\mathcal{P}(\mathcal{H})$ the set of all orthogonal projections and $\mathcal{S}(\mathcal{H})$ the set of all density operators, i.e., the trace class positive operators on $\mathcal{H}$ of unit trace. The elements $\rho \in \mathcal{S}(\mathcal{H})$ represent the states of a quantum system, and the probability that a quantum effect A occurs in the state ρ is given by $P_\rho(A) = \operatorname{tr}(\rho A)$. If A is a quantum observable, then we define

$$|A| = (A^2)^{\frac{1}{2}}, \qquad A^+ = \frac{|A| + A}{2} \quad \text{and} \quad A^- = \frac{|A| - A}{2},$$

where $(A^2)^{\frac{1}{2}}$ is the unique positive square root of A^2. For $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, the trace distance and the fidelity are defined respectively by

$$D(\rho,\sigma) = \frac{1}{2}\operatorname{tr}|\rho - \sigma| \quad \text{and} \quad F(\rho,\sigma) = \Big(\operatorname{tr}\sqrt{\rho^{\frac{1}{2}}\sigma\rho^{\frac{1}{2}}}\Big)^2.$$

In [12], a new fidelity, called superfidelity, was defined as

$$G(\rho,\sigma) = \operatorname{tr}\rho\sigma + \sqrt{1-\operatorname{tr}\rho^2}\sqrt{1-\operatorname{tr}\sigma^2}.$$

It was shown that when ρ and σ are qubit states, the superfidelity $G(\rho,\sigma)$ coincides with fidelity $F(\rho,\sigma)$. The most interesting feature of superfidelity is that it gives an upper bound for fidelity [12], that is

$$F(\rho,\sigma) \leqslant G(\rho,\sigma),$$

for $\rho,\sigma \in \mathcal{S}(\mathcal{H})$, where $\mathcal{H}$ is a finite-dimensional space. Very recently, the properties of superfidelity have been established and studied by many authors [6,12–16]. In [16], the superfidelity has been applied to quantum circuits to enable a measure of distance between two quantum channels, and in [15] a new probability measure was introduced in terms of superfidelity. In particular, if $\mathcal{H}$ is a finite-dimensional space, the inequality

$$1 - G(\rho,\sigma) \leqslant D(\rho,\sigma) \tag{1}$$

was obtained in [14]. This inequality was also conjectured in [13] and verified numerically for small dimensions.

The purpose of this Letter is to consider inequality (1) in an infinite-dimensional space and give a necessary and sufficient condition for saturation of inequality (1). In addition, we consider the perturbational upper and lower bound of $G(\phi_{\mathcal{A}}(\rho),\rho)$ over all states of a quantum system, where $\phi_{\mathcal{A}}$ is a trace preserving, unital quantum operation.

2. Saturation of inequality (1)

The following result is well known. For the reader's convenience, we give a simple proof.

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Lemma 2.1. *Let $\rho, \sigma \in S(\mathcal{H})$. Then $0 \leqslant D(\rho,\sigma) \leqslant 1$, and $D(\rho,\sigma) = 1$ if and only if $\rho\sigma = 0$.*

Proof. As in the proof for the finite-dimensional case, we get $D(\rho,\sigma) = \max\{\mathrm{tr}(P(\rho-\sigma)):\ P \in P(\mathcal{H})\}$. Thus $0 \leqslant D(\rho,\sigma) \leqslant 1$.

If $D(\rho,\sigma) = 1$, then there exists a projection $P_1 \in P(\mathcal{H})$ such that $\mathrm{tr}(P_1\rho) - \mathrm{tr}(P_1\sigma) = 1$. It follows from $0 \leqslant \mathrm{tr}(P_1\rho) \leqslant 1$ and $0 \leqslant \mathrm{tr}(P_1\sigma) \leqslant 1$ that $\mathrm{tr}(P_1\rho) = 1$ and $\mathrm{tr}(P_1\sigma) = 0$, which yields $P_1\sigma = 0$ and $\rho P_1 = \rho$. Thus $\rho\sigma = \rho P_1 \sigma = 0$. The inverse implication is evident. □

The following is the main result of this section.

Theorem 2.2. *Let $\rho, \sigma \in S(\mathcal{H})$. Then $1 - D(\rho,\sigma) \leqslant G(\rho,\sigma)$, with equality if and only if one of the following hypotheses holds:*

(i) $\rho = \sigma$;
(ii) $\rho\sigma = \sigma\rho$ *and* ρ *or* σ *is a pure state.*

Proof. Let P_+ denote the projection onto the subspace $\overline{R[(\rho-\sigma)^+]}$ and $P_- = I - P_+$, where $\overline{R[(\rho-\sigma)^+]}$ is the closure of the range $(\rho-\sigma)^+$. Then $P_+(\rho-\sigma) = (\rho-\sigma)^+$, and $P_-(\rho-\sigma) = -(\rho-\sigma)^-$. By a similar calculation as in [14], we obtain that the inequalities (21)–(24) and Eq. (25) in [14] also hold in an infinite-dimensional system (for details, see [14, Theorem 1]). Thus the inequality $1 - D(\rho,\sigma) \leqslant G(\rho,\sigma)$ holds in an infinite-dimensional system as well.

In the following, we show a necessary and sufficient condition for the equality $1 - D(\rho,\sigma) = G(\rho,\sigma)$.

The "if" part: If $\rho = \sigma$, then it is clear that $1 - D(\rho,\sigma) = G(\rho,\sigma)$. In the following, we assume that $\rho\sigma = \sigma\rho$ and ρ is a pure state. Suppose $\rho = |\phi\rangle\langle\phi|$ and the subspace $\mathcal{H}_1$ is spanned by the state $|\phi\rangle$. Then

$$\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_1^{\perp} \to \mathcal{H}_1 \oplus \mathcal{H}_1^{\perp}. \tag{2}$$

Let

$$\sigma = \begin{pmatrix} a_1 & \sigma_1 \\ \sigma_1^* & \sigma_2 \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_1^{\perp} \to \mathcal{H}_1 \oplus \mathcal{H}_1^{\perp}, \tag{3}$$

where σ_1 is an operator from $\mathcal{H}_1^{\perp}$ into $\mathcal{H}_1$ and σ_2 is an operator from $\mathcal{H}_1^{\perp}$ into $\mathcal{H}_1^{\perp}$. It is obvious that $\sigma \in S(\mathcal{H})$ implies that $a_1 \geqslant 0$, $\sigma_2 \geqslant 0$, and $a_1 + \mathrm{tr}(\sigma_2) = 1$. By a direct calculation, we obtain from $\rho\sigma = \sigma\rho$ that $\sigma_1 = 0$ and $\sigma_1^* = 0$. Thus

$$|\rho - \sigma| = \begin{pmatrix} 1 - a_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_1^{\perp} \to \mathcal{H}_1 \oplus \mathcal{H}_1^{\perp},$$

which means $D(\rho,\sigma) = 1 - a_1$. By Eqs. (2) and (3), we have $\mathrm{tr}(\rho\sigma) = a_1$, so $1 - D(\rho,\sigma) = G(\rho,\sigma)$.

The "only if" part: As in the proof of [14, Eq. (25)] for the finite-dimensional case, we have

$$\begin{aligned} D(\rho,\sigma) &= \frac{1}{2}(\mathrm{tr}\, P_+\rho - \mathrm{tr}\, P_+\sigma + \mathrm{tr}\, P_-\sigma - \mathrm{tr}\, P_-\rho) \\ &= 1 - \mathrm{tr}\, P_+\sigma - \mathrm{tr}\, P_-\rho. \end{aligned} \tag{4}$$

By the assumption that $1 - D(\rho,\sigma) = G(\rho,\sigma)$, we have (for a detailed calculation see [14, Eq. (26)])

$$\sqrt{1 - \mathrm{tr}\,\rho^2}\sqrt{1 - \mathrm{tr}\,\sigma^2} = \mathrm{tr}\, P_+(I-\rho)\sigma + \mathrm{tr}\, P_-\rho(I-\sigma). \tag{5}$$

It is clear that $\mathrm{tr}[P_+(I-\rho)(\rho-\sigma)] = \mathrm{tr}[(I-\rho)(\rho-\sigma)^+] \geqslant 0$ implies

$$\mathrm{tr}\big[P_+(I-\rho)\rho\big] \geqslant \mathrm{tr}\big[P_+(I-\rho)\sigma\big]. \tag{6}$$

Similarly, we obtain that

$$\mathrm{tr}\big[P_-\rho(I-\rho)\big] \geqslant \mathrm{tr}\big[P_-\rho(I-\sigma)\big]. \tag{7}$$

Furthermore, as in the finite-dimensional case [14, inequalities (22), (23)], we also get

$$1 - \mathrm{tr}\,\rho^2 \geqslant \mathrm{tr}\big[P_+(I-\rho)\sigma\big] + \mathrm{tr}\big[P_-\rho(I-\sigma)\big] \tag{8}$$

and

$$1 - \mathrm{tr}\,\sigma^2 \geqslant \mathrm{tr}\big[P_+(I-\rho)\sigma\big] + \mathrm{tr}\big[P_-\rho(I-\sigma)\big]. \tag{9}$$

Case 1. Suppose that $1 - \mathrm{tr}\,\sigma^2 > 0$ and $1 - \mathrm{tr}\,\rho^2 > 0$. Then combining (5), (8) and (9), we have

$$1 - \mathrm{tr}\,\rho^2 = \mathrm{tr}\big[P_+(I-\rho)\sigma\big] + \mathrm{tr}\big[P_-\rho(I-\sigma)\big] = 1 - \mathrm{tr}\,\sigma^2, \tag{10}$$

which means $G(\rho,\sigma) = \mathrm{tr}\,\rho\sigma + 1 - \mathrm{tr}\,\rho^2$, so $1 - D(\rho,\sigma) = G(\rho,\sigma)$ implies that $\mathrm{tr}\,|\rho-\sigma| + 2\,\mathrm{tr}\,\rho\sigma - 2\,\mathrm{tr}\,\rho^2 = 0$. Thus $\mathrm{tr}[|\rho-\sigma| - (\rho-\sigma)^2] = 0$.

On the other hand, as $-I \leqslant \rho - \sigma \leqslant I$, we have $|\rho-\sigma| \leqslant I$, so $(\rho-\sigma)^2 = |\rho-\sigma|^2 \leqslant |\rho-\sigma|$. Then $|\rho-\sigma| = |\rho-\sigma|^2$, which means that $|\rho-\sigma|$ is a projection. Hence $(\rho-\sigma)^+$ and $(\rho-\sigma)^-$ are projections. By Lemma 2.1, we know that

$$1 \geqslant D(\rho,\sigma) = \mathrm{tr}(\rho-\sigma)^+ = \mathrm{tr}(\rho-\sigma)^-;$$

hence we conclude that

$$\mathrm{tr}(\rho-\sigma)^+ = \mathrm{tr}(\rho-\sigma)^- = 0 \quad \text{or}$$
$$\mathrm{tr}(\rho-\sigma)^+ = \mathrm{tr}(\rho-\sigma)^- = 1.$$

Thus $D(\rho,\sigma) = 0$ or $D(\rho,\sigma) = 1$, so Lemma 2.1 implies that $\rho = \sigma$ or $\rho\sigma = 0$.

Moreover, if $\rho\sigma = 0$, then it is clear that $1 - D(\rho,\sigma) = G(\rho,\sigma)$ implies $\sqrt{1 - \mathrm{tr}\,\rho^2}\sqrt{1 - \mathrm{tr}\,\sigma^2} = 0$, so $1 - \mathrm{tr}\,\rho^2 = 0$ or $1 - \mathrm{tr}\,\sigma^2 = 0$. This contradiction with the assumption that $1 - \mathrm{tr}\,\sigma^2 > 0$ and $1 - \mathrm{tr}\,\rho^2 > 0$ shows that $\rho = \sigma$.

Case 2. Suppose that $1 - \mathrm{tr}\,\sigma^2 = 0$ or $1 - \mathrm{tr}\,\rho^2 = 0$. Without loss of generality, we assume that $1 - \mathrm{tr}\,\rho^2 = 0$; then $\mathrm{tr}\,\rho^2 = 1$ implies that ρ is a pure state. Furthermore, combining (5) and (8), we get

$$1 - \mathrm{tr}\,\rho^2 = \mathrm{tr}\big[P_+(I-\rho)\sigma\big] + \mathrm{tr}\big[P_-\rho(I-\sigma)\big].$$

Combining (6) and (7), we have

$$\mathrm{tr}\big[P_+(I-\rho)\rho\big] = \mathrm{tr}\big[P_+(I-\rho)\sigma\big]$$

and

$$\mathrm{tr}\big[P_-\rho(I-\rho)\big] = \mathrm{tr}\big[P_-\rho(I-\sigma)\big],$$

which yields

$$(I-\rho)(\rho-\sigma)^+ = 0$$

and

$$\rho(\rho-\sigma)^- = 0.$$

Then

$$(\rho-\sigma)^+ - \rho(\rho-\sigma) = (I-\rho)(\rho-\sigma)^+ + \rho(\rho-\sigma)^- = 0,$$

which means

$$\rho(\rho-\sigma) = (\rho-\sigma)^+ = (\rho-\sigma)\rho,$$

so $\rho\sigma = \sigma\rho$. □

Remark. Theorem 2.2 shows that the equation $1 - D(\rho,\sigma) = G(\rho,\sigma)$ implies the equation $F(\rho,\sigma) = G(\rho,\sigma)$. However, the reverse implication does not hold. A simple example comes from the two-dimensional Hilbert space, in which for any two states the superfidelity is equal to quantum fidelity [12].

For superfidelity, we can define the following function:

$$C(\rho,\sigma) := \sqrt{1 - G(\rho,\sigma)}, \quad \text{for } \rho,\sigma \in S(\mathcal{H}).$$

We recall that the Hilbert–Schmidt norm of $\rho \in S(\mathcal{H})$ is defined as $\|\rho\|_{\mathrm{HS}} := (\operatorname{tr}\rho^2)^{\frac{1}{2}}$. It is shown in [13] that $C(\rho,\sigma)$ is a genuine metric. The following two results show that the topologies induced by $C(\rho,\sigma)$ and $D_{\mathrm{HS}}(\rho,\sigma)$ are the same on $S(\mathcal{H})$ if $\mathcal{H}$ is a finite-dimensional space.

Proposition 2.3. *Let $\rho,\sigma \in S(\mathcal{H})$. Then*

(i) $\|\rho - \sigma\|_{\mathrm{HS}} \leqslant \sqrt{2}C(\rho,\sigma)$,
(ii) $\|\rho - \sigma\|_{\mathrm{HS}} = \sqrt{2}C(\rho,\sigma)$ *if and only if* $\operatorname{tr}\rho^2 = \operatorname{tr}\sigma^2$.

Proof. (i)

$$\|\rho - \sigma\|_{\mathrm{HS}} \leqslant \sqrt{2}\sqrt{1 - G(\rho,\sigma)}$$
$$\iff \quad 2 - 2\sqrt{1 - \operatorname{tr}\rho^2}\sqrt{1 - \operatorname{tr}\sigma^2} \geqslant \operatorname{tr}\rho^2 + \operatorname{tr}\sigma^2$$
$$\iff \quad \left(1 - \operatorname{tr}\rho^2\right) + \left(1 - \operatorname{tr}\sigma^2\right) \geqslant 2\sqrt{1 - \operatorname{tr}\rho^2}\sqrt{1 - \operatorname{tr}\sigma^2}.$$

(ii) The conclusion is clear, because the equality between geometric and arithmetic means is satisfied only in the case of equal factors in the proof of (i). □

Proposition 2.4. *Let $\mathcal{H}$ be a finite-dimensional space. Then the topology induced by $C(\rho,\sigma)$ and $D_{\mathrm{HS}}(\rho,\sigma)$ is the same on $S(\mathcal{H})$, where $D_{\mathrm{HS}}(\rho,\sigma) := \|\rho - \sigma\|_{\mathrm{HS}}$ is called the Hilbert–Schmidt distance.*

Proof. It is sufficient to show that

$$C(\rho_n,\rho) \longrightarrow 0 \quad \iff \quad D_{\mathrm{HS}}(\rho_n,\rho) \longrightarrow 0, \quad n \longrightarrow \infty.$$

Given Proposition 2.3, it remains to show that if $D_{\mathrm{HS}}(\rho_n,\rho) \longrightarrow 0$, then $C(\rho_n,\rho) \longrightarrow 0$, $n \longrightarrow \infty$. Assume that $D_{\mathrm{HS}}(\rho_n,\rho) \longrightarrow 0$, so $\sqrt{\operatorname{tr}[(\rho_n - \rho)^2]} \longrightarrow 0$, $n \longrightarrow \infty$. Using the Cauchy–Schwarz inequality, we get $\operatorname{tr}(|\rho_n - \rho|) \leqslant \sqrt{\operatorname{tr}[(\rho_n - \rho)^2]}\sqrt{m}$, where m is the dimension of $\mathcal{H}$, which yields $\operatorname{tr}(|\rho_n - \rho|) \longrightarrow 0$, $n \longrightarrow \infty$. Thus by the formula $C(\rho,\sigma) \leqslant \sqrt{D(\rho,\sigma)}$, we know that

$$C(\rho_n,\rho) \leqslant \sqrt{\frac{1}{2}\operatorname{tr}\left(|\rho_n - \rho|\right)} \longrightarrow 0, \quad \text{for } n \longrightarrow \infty. \qquad \square$$

3. Characterization of $\inf\{G(\phi_{\mathcal{A}}(\rho),\rho)\colon \rho \in S(\mathcal{H})\}$

In this section, let $\mathcal{B}(\mathcal{H})$ be the set of all bounded linear operators on a finite-dimensional Hilbert space $\mathcal{H}$. Recall that the operator-sum representation is a key result of the quantum operation formalism. Namely, according to a well-known result [17], a general *completely positive* quantum operation is a bounded linear operator defined on $\mathcal{B}(\mathcal{H})$ which has the form

$$\phi_{\mathcal{A}}(B) = \sum_{i=1}^{n} A_i B A_i^*, \tag{11}$$

where $A_i \in \mathcal{B}(\mathcal{H})$ $(1 \leqslant i \leqslant n)$ are arbitrary bounded operators. If $\sum_{i=1}^{n} A_i^* A_i = I$, then $\phi_{\mathcal{A}}$ is said to be *trace preserving*. If $\phi_{\mathcal{A}}(I) = I$ (equivalently $\sum_{i=1}^{n} A_i A_i^* = I$), then the quantum operation $\phi_{\mathcal{A}}$ is called *unital*.

Quantum measurements that have more than two values are described by quantum effect valued measures, that is, for $i = 1,2,\ldots,n$, $0 \leqslant A_i \leqslant I$ satisfy $\sum_{i=1}^{n} A_i = I$. In this case, the *Lüders operation* (see [18]) is defined as a bounded linear map $\Lambda_{\mathcal{A}} : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ that satisfies

$$\Lambda_{\mathcal{A}}(S) = \sum_{i=1}^{n} A_i^{\frac{1}{2}} S A_i^{\frac{1}{2}}.$$

In the following, we consider the perturbational upper and lower bound of $G(\phi_{\mathcal{A}}(\rho),\rho)$ over all states of a quantum system.

Theorem 3.1. *Let $\phi_{\mathcal{A}}$ be a trace preserving, unital quantum operation. Then*

(i) $\sup\{G(\phi_{\mathcal{A}}(\rho),\rho)\colon \rho \in S(\mathcal{H})\} = 1$,
(ii) $\inf\{G(\phi_{\mathcal{A}}(\rho),\rho)\colon \rho \in S(\mathcal{H})\} = 0$ *if and only if there exists* $|\phi\rangle \in \mathcal{H}$, *such that* $\operatorname{tr}(A_i|\phi\rangle\langle\phi|) = 0$, *for* $i = 1,2,\ldots,n$.

Proof. Since $\mathcal{H}$ is a finite-dimensional Hilbert space, we know that $S(\mathcal{H})$ is a compact set in the norm topology. It is clear that the function $G(\phi_{\mathcal{A}}(\rho),\rho)$ of ρ is a continuous function in the norm topology. Thus sup and inf can be replaced by max and min, respectively.

(i) By Schauder's fixed point theorem [19, p. 150], we get that there exists a state σ such that $\phi_{\mathcal{A}}(\sigma) = \sigma$, as $S(\mathcal{H})$ is a compact convex set. Then it is easy to see that $\sup\{G(\phi_{\mathcal{A}}(\rho),\rho)\colon \rho \in S(\mathcal{H})\} = 1$.
(ii) If $\inf\{G(\phi_{\mathcal{A}}(\rho),\rho)\colon \rho \in S(\mathcal{H})\} = 0$, then there exists $\rho_0 \in S(\mathcal{H})$, such that

$$\phi_{\mathcal{A}}(\rho_0)\rho_0 = 0 \quad \text{and} \quad \sqrt{1 - \operatorname{tr}\rho_0^2}\sqrt{1 - \operatorname{tr}\left[\phi_{\mathcal{A}}(\rho_0)^2\right]} = 0.$$

Case 1. If $1 - \operatorname{tr}\rho_0^2 = 0$, then ρ_0 is a pure state. Let $\rho_0 = |\phi\rangle\langle\phi|$. It follows from $\phi_{\mathcal{A}}(\rho_0)\rho_0 = 0$ that

$$\sum_{i=1}^{n}\left(\langle\phi|A_i^*|\phi\rangle A_i\right)|\phi\rangle\langle\phi| = \sum_{i=1}^{n}\left(A_i|\phi\rangle\langle\phi|A_i^*\right)|\phi\rangle\langle\phi| = 0,$$

so

$$\sum_{i=1}^{n}\left(\left|\langle\phi|A_i|\phi\rangle\right|^2\right) = 0,$$

which means $\langle\phi|A_i|\phi\rangle = 0$, for $i = 1,2,\ldots,n$. Hence $\operatorname{tr}(A_i|\phi\rangle\langle\phi|) = 0$, for $i = 1,2,\ldots,n$.

Case 2. If $1 - \operatorname{tr}[\phi_{\mathcal{A}}(\rho_0)^2] = 0$, then $\phi_{\mathcal{A}}(\rho_0)$ is a pure state. Let $\phi_{\mathcal{A}}(\rho_0) = |\psi\rangle\langle\psi|$. It is easy to see that

$$1 = \langle\psi|\sum_{i=1}^{n} A_i\rho_0 A_i^*|\psi\rangle = \sum_{i=1}^{n}\langle\psi|A_i\rho_0 A_i^*|\psi\rangle$$
$$= \sum_{i=1}^{n}\left\|\rho_0^{\frac{1}{2}}A_i^*|\psi\rangle\right\|^2 \leqslant \left(\sum_{i=1}^{n}\left\|A_i^*|\psi\rangle\right\|^2\right)\left\|\rho_0^{\frac{1}{2}}\right\|^2.$$

As

$$\sum_{i=1}^{n}\left\|A_i^*|\psi\rangle\right\|^2 = \langle\psi|\sum_{i=1}^{n} A_i A_i^*|\psi\rangle = 1,$$

we have $\|\rho_0\| = \|\rho_0^{\frac{1}{2}}\|^2 \geqslant 1$, then $\operatorname{tr}\rho_0 = 1$ implies that ρ_0 is a pure state. The remaining part of the proof is the same as in Case 1. □

Corollary 3.2. *Let $\Lambda_{\mathcal{A}}$ be a Lüders operation. Then* $\inf\{G(\Lambda_{\mathcal{A}}(\rho), \rho): \rho \in S(\mathcal{H})\} > 0$.

Proof. Assume the opposite, i.e., that

$$\inf\{G(\Lambda_{\mathcal{A}}(\rho), \rho): \rho \in S(\mathcal{H})\} = 0.$$

By Theorem 3.1, we know that there exists $|\phi\rangle \in \mathcal{H}$, such that $\mathrm{tr}(A_i|\phi\rangle\langle\phi|) = 0$, for $i = 1, 2, \ldots, n$, so $A_i|\phi\rangle\langle\phi| = 0$. Then $A_i^2|\phi\rangle\langle\phi| = 0$, for $i = 1, 2, \ldots, n$, which yields

$$|\phi\rangle\langle\phi| = \sum_{i=1}^{n} A_i^2|\phi\rangle\langle\phi| = 0.$$

This is a contradiction. □

4. Conclusion

As an extension of quantum fidelity, a new fidelity, called superfidelity is defined in [12]. In this Letter, we consider an important inequality involving superfidelity and trace metric in an infinite-dimensional space and give a necessary and sufficient condition for saturation of this inequality. Our main result shows that this inequality is saturated only in an extreme condition. On the other hand, *in the finite-dimensional case* we also obtain a topology structure involving superfidelity and the upper and lower perturbation bound between a state and its transformation by a quantum operation. All results in this Letter show that superfidelity can be used to infer the distinguishability of states.

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