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New solutions for ordinary differential equations

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ABSTRACT

This paper introduces a new method for solving ordinary differential equations (ODEs) that enhances existing methods that are primarily based on finding integrating factors and/or point symmetries. The starting point of the new method is to find a non-invertible mapping that maps a given ODE to a related higher-order ODE that has an easily obtained integrating factor. As a consequence, the related higher-order ODE is integrated. Fixing the constant of integration, one then uses existing methods to solve the integrated ODE. By construction, each solution of the integrated ODE yields a solution of the given ODE. Moreover, it is shown when the general solution of an integrated ODE yields either the general solution or a family of particular solutions of the given ODE. As an example, new solutions are obtained for an important class of nonlinear oscillator equations. All solutions presented in this paper cannot be obtained using the current MAPLE ODE solver.

1. Introduction

In the late 19th century, Sophus Lie, inspired by the Galois theory for algebraic equations, presented an ingenious approach for solving ordinary differential equations (ODEs) (Olver, 1986; Bluman and Kumei, 1989; Bluman and Anco, 2002). Since Lie's work, a common belief is that reduction by Lie symmetries (point or contact) is *the* method for solving ODEs. It is also the basis for the core algorithm of current ODE solvers such as the MAPLE solver dsolve.

Consider the nonlinear oscillator

$$u'' + u^2u' + 2u^4 - 18u^3 = 0. \tag{1}$$

ODE (1) is a special case of a more general situation discussed in Case II in Section 5 of this paper. One can easily show that the translations $x \to x + \varepsilon$ yield the only point symmetry of ODE (1). The

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corresponding invariant solutions are the obvious solutions $u(x) = 0$ and $u(x) = 9$. On the other hand, if one uses the translation point symmetry to reduce the order of ODE (1), then one obtains the complicated Abel equation

$$\left(w'(t) + t^2\right) w(t) + 2t^4 - 18t^3 = 0, \tag{2}$$

where $t = u$ and $w = u'$. ODE (2) has no obvious reduction.

Now, substitute $u(x) = v'(x)$ in (1) to obtain the third-order ODE

$$v''' + v'^2 v'' + 2v'^4 - 18v'^3 = 0. \tag{3}$$

By standard methods, one can easily find the integrating factor $\mu[V] = e^{6V}$, where V is an arbitrary function of x. This yields the first integral of (3) given by

$$G[v] = \left(v'' + \frac{v'^3}{3} - 3v'^2\right) e^{6v} = c. \tag{4}$$

Then through the mapping $u = v'$, any solution of $G[v] = c$, where c is an arbitrary constant, is necessarily a solution of (1), but the converse is not true. Now, restrict the mapping to $G[v] = c$. Then, the mapping

$$u = v', \quad u' = v'' = -\frac{1}{3}v'^3 + 3v'^2 - ce^{-6v} \tag{5}$$

is invertible if and only if the right-hand side of the second equation of (5) depends explicitly on v, i.e., $c \neq 0$. In this case, the differential equation $G[v] = c$ is more complicated than our given Eq. (1). However, when $c = 0$, precisely when the mapping becomes non-invertible, the equation $G = 0$ is easily solved, and its general solution yields the one-parameter family of solutions

$$u(x) = \frac{9}{1 + \mathcal{W}(c_1 e^{-27x})} \tag{6}$$

for the nonlinear oscillator (1). (In (6), $\mathcal{W}(x)$ is the Lambert W function defined to be the inverse of the function $w \to we^w$ and c_1 is an arbitrary constant.) The way we obtained solution (6) to ODE (1) is naively the method we are presenting in this paper, which extends the method introduced in Bluman and Reid (1988).

The question which arises naturally after this example is the following. Why is the one-parameter family of solutions (6) of ODE (1) not captured using Lie's approach? The answer is simple: Lie's approach is based on symmetries which act on the *whole* solution set of a given ODE. In particular, in Lie's approach, one assumes that

$$X(F[u]) = 0 \mod F[u] = 0, \tag{7}$$

where X is the infinitesimal generator of a point (or, more generally, contact) symmetry, and $F[u]$, in our example, is the left-hand side of (1). For ODE (1), we were unable to find any such symmetries other than the translations $x \to x + \varepsilon$. On the other hand, the above non-invertible mapping (5) with $c = 0$ induces for ODE (1) the "symmetry" $X = (9x - v)\,\partial_x + u\,(u - 9)\,\partial_u$, where v is such that $u - v' = 0$,[2] by push forwarding one of the eight point symmetries of $G[v] = 0$. When applied to $F[u]$, instead of $X(F[u]) = 0 \mod F[u] = 0$, one has

$$X(F[u]) = 0 \mod I[u] = 0, \tag{8}$$

2 If $\begin{array}{c} \mathcal{S} \hookrightarrow J^2(\mathbb{C},\mathbb{C}^2) \\ \searrow\swarrow \\ J^0(\mathbb{C},\mathbb{C}^2) \end{array}$ denotes the affine subbundle of the 2-jet space $J^2(\mathbb{C},\mathbb{C}^2)$ defined by the ODE system $F[u] = 0$ and $u - v' = 0$, then the vector field X lives in $\Gamma TJ^0(\mathbb{C},\mathbb{C}^2)$. It can be lifted uniquely to $J^2(\mathbb{C},\mathbb{C}^2)$ by preserving its contact structure (and that of $J^1(\mathbb{C},\mathbb{C}^2)$). The resulting vector field, called the second prolongation of X, is also denoted by X. In particular, in (8), $X \in \Gamma TJ^2(\mathbb{C},\mathbb{C}^2)$).

where

$$I[u] := u' + \frac{1}{3}u^3 - 3u^2. \tag{9}$$

The one-parameter family of solutions (6) is the general solution of $I[u] = 0$, as can be seen from (5), when $c = 0$. The vector field X leaves only the subset of solutions $I[u] = 0$ invariant, not the whole set $F[u] = 0$. This aspect will be discussed in depth in a second paper (Bluman and Dridi, in preparation).

2. Preliminaries

Consider a given nth-order algebraic differential equation

$$F(x, u, u', \ldots, u^{(n)}) = 0, \tag{10}$$

i.e., the corresponding *ODE function* $F[U] := F(x, U, U', \ldots, U^{(n)})$ is a polynomial in the variables $U, U', \ldots$, and $U^{(n)}$, where U is an arbitrary function of x. The coefficient field is the field of rational functions $\mathbb{C}(x)$. The notation for this is $F[U] \in \mathbb{C}(x)[U, U', \ldots, U^{(n)}]$. There is a clear distinction between the ODE $F[u] = 0$ and the ODE function $F[U]$. We assume that $F[U]$ is linear in the highest derivative $U^{(n)}$.

Consider the mapping

$$U = f[V] = f(x, V, V') \in \mathbb{C}(x)(V, V') \tag{11}$$

rational in V and V'. We assume that $f[V]$ has an essential dependence on V'. Let

$$\mu[V] = \mu(x, V, V', \ldots, V^{(\ell)}), \quad \ell \leq n, \tag{12}$$

be a multiplier function. Let

$$H[V] = H\left(x, V, V', \ldots, V^{(n+1)}\right) \in \mathbb{C}(x)\left(V, V', \ldots, V^{(n+1)}\right) \tag{13}$$

be the rational function obtained after substituting the mapping (11) into the ODE function $F[U]$. In particular,

$$H[V] = H\left(x, V, V', \ldots, V^{(n+1)}\right) = F\left(x, f[V], \frac{df[V]}{dx}, \ldots, \frac{d^{(n)}f[V]}{dx^n}\right), \tag{14}$$

where $\frac{d}{dx} := \partial_x + V'\partial_V + \cdots V^{(n+2)}\partial_{V^{(n+1)}}$. Now, suppose that the multiplier (12) is an *integrating factor* for (14), i.e.,

$$\mu[V]H[V] = \frac{d}{dx}G[V], \tag{15}$$

for some function

$$G[V] = G\left(x, V, V', \ldots, V^{(n)}\right). \tag{16}$$

The function $\mu[V]$ in (12) is an integrating factor for (14) if and only if

$$E_V\left(\mu[V]H[V]\right) \equiv 0 \tag{17}$$

in terms of the Euler operator

$$E_V = \frac{\partial}{\partial V} - \frac{d}{dx}\frac{\partial}{\partial V'} + \cdots + (-1)^{(n+1)}\frac{d^{n+1}}{dx^{n+1}}\frac{\partial}{\partial V^{(n+1)}}. \tag{18}$$

The left-hand side of (17) is linear in $\mu[V]$ and its partial derivatives with coefficients in the field $\mathbb{C}(x, V, V', \ldots, V^{(n+2)})(f, f_x, f_V, f_{V'}, \ldots)$. Now, suppose that $v(x) = \Theta(x)$ is a solution of the *integrated ODE*

$$G[v] = G\left(x, v, v', \ldots, v^{(n)}\right) = c, \tag{19}$$

for some fixed constant c. Then, through the mapping (11), it follows that

$$u = f[v] = f\left(x, \Theta(x), \Theta'(x)\right) \tag{20}$$

is a solution of the given ODE (10). Moreover, if $v(x) = \Phi(x; c_1, \dots, c_n)$ is a general solution of the integrated ODE (19), then

$$u = f\left(x, \Phi(x; c_1, \dots, c_n), \Phi_x(x; c_1, \dots, c_n)\right) \tag{21}$$

solves the given ODE (10). If $c_1, \dots, c_n$ are n essential constants in solution (21), then (21) yields a general solution of the given ODE (10).

Proposition 1. *A general solution of the integrated ODE* (19)*, for a fixed value of the constant c, yields a general solution of the given ODE* (10) *if and only if* $u = f(x, v, v')$ *and* $u' = \frac{d}{dx}f(x, v, v')$ *are functionally independent over* $\mathbb{C}(x)$ *modulo the corresponding integrated ODE* (19)*, i.e., if and only if the determinant*

$$\det\begin{pmatrix} \partial_v f[v] & \partial_{v'} f[v] \\ \partial_v \frac{d}{dx} f[v] & \partial_{v'} \frac{d}{dx} f[v] \end{pmatrix} \neq 0 \mod G[v] = c. \tag{22}$$

Proof. The transverse mapping f is by construction a contact preserving mapping, i.e., it sends the contact distribution of Eq. (10) to the contact distribution of Eq. (13). The restriction to Eq. (19), i.e., composing the mapping f with a contact preserving surjective map, is again a contact preserving map. The condition (22) makes the restricted f a diffeomorphism of $J^n(\mathbb{C}, \mathbb{C})$, and hence a contact transformation of $J^n(\mathbb{C}, \mathbb{C})$. □

To illustrate the above proposition, consider again the example above, where Eqs. (10) and (19) correspond to the second-order ODEs ($n = 2$) given by (1) and (4). The restriction of the mapping $u = v'$ to Eq. (4) is given by

$$u = v', \quad u' = v'' = -\frac{1}{3}v'^3 + 3v'^2 - ce^{-6v}. \tag{23}$$

Here, the determinant in (22) is equal to $-6ce^{-6v}$. The mapping (23) is a contact transformation relating Eqs. (10) and (19) if and only if c is not zero.

3. Outline of the method

The method introduced in Section 2 involves three procedures. The first procedure, lie_solution, computes a particular solution of partial differential equation (PDE) systems using Lie symmetries. The second procedure, solution_pair, looks for a pair $(f[V], \mu[V])$ satisfying the nonlinear PDE system (17) with the inequations $f_{V'} \neq 0$, $\mu \neq 0$. The third and main procedure is the solver.

3.1. *The first procedure,* lie_solution

Given a system of differential equations and inequations, the procedure lie_solution computes a *particular* (i.e., invariant) solution (one or more) of the given system using Lie symmetries. This procedure is very similar to the Maple command symgen. The inequations are handled by localization (for instance, $f_{V'} \neq 0$ is replaced by $f_{V'} w = 1$).

3.2. *The second procedure,* solution_pair

Let $k := \mathbb{C}(x, V, V', \dots, V^{(n-1)})$ be the differential field of rational functions endowed with the derivations $\{\partial_x, \partial_V, \partial_{V'}, \dots, \partial_{V^{(n-1)}}\}$. To solve the nonlinear system (17) with the inequations $f_{V'} \neq 0$, $\mu \neq 0$, we use a table of ansätze. Each entry of the table corresponds to a differential system $\Sigma \subset k$. For instance (in the case of second-order ODEs), the first entry is given by the list $[f_x, f_V, \mu_x, \mu_{V'}, \mu_{V''}]$ corresponding to the assumptions that $f[V]$ is a function of v' and $\mu[V]$ a function of V only.

To find a solution pair, we proceed as follows.

PROCEDURE solution_pair
Input *An ODE function* $F[U]$
Output *A solution pair* $(f[V], \mu[V])$ *satisfying* (17) *with* $f_{V'} \neq 0$, $\mu \neq 0$
(1) for *each* Σ *in the table of ansätze* do
(i) *compute a characteristic set* C *of the system* (17) *union* Σ *with* $f_{V'} \neq 0$, $\mu \neq 0$
(ii) return $(f[V], \mu[V]) :=$ lie_solution(C) *if not empty*

3.3. *The solver* ode_solver

PROCEDURE ode_solver
Input *An ODE* $F[u] = 0$
Output *Solution of* $F[u] = 0$
(1) $(f[V], \mu[V]) :=$ solution_pair$(F[U])$
(2) *compute a characteristic set* C *of the linear system* (15) *with* $G[V]_{V^{(n-1)}} \neq 0$
(3) $G[V] :=$ lie_solution(C)
(4) *solve the ODE system* $\{G[v] - c(x) = 0, c'(x) = 0\}$
(5) return $u = f[\Theta(x)]$ *for each solution* $(v = \Theta(x), c = c_0)$

Remark 1. In step (4), the solving process of the ODE system $\{G[v] - c(x) = 0, c'(x) = 0\}$ starts by computing first integrals using lie_solve. If n functionally independent first integrals for the same value of $c = c_0$ are found, then the general solution of the integrated ODE $G[v] - c_0 = 0$ is obtained. If no such first integrals are found, then the solving process returns a list of invariant solutions.

The above method is now illustrated through two examples (a more complicated example is the nonlinear oscillator treated in Section 5).

Example. Consider again ODE (1), i.e.,

$$u'' + u^2u' + 2u^4 - 18u^3 = 0.$$

The corresponding ODE function is given by $F[U] = U'' + U^2U' + 2U^4 - 18U^3$. Restrict the mapping (11) to be of the form $U = f[V] = \psi(V)V'$. For the multiplier (15), consider the ansatz $\mu[V] = \mu(V)$. For the corresponding determining equation (17), one obtains the solution pair

$$(f[V], \mu[V]) = \left(\psi(V)V', c_1e^{\int 6\psi(V)dV}\right),$$

where $\psi(V)$ is an arbitrary function. Setting $\psi(V) = c_1 = 1$, one obtains the third-order ODE

$$H[v] := v''' + {v'}^2v'' + 2{v'}^4 - 18{v'}^3 = 0,$$

which in turn yields the integrated ODE (4). The rest of the calculation is already given in the introduction.

Now, consider the geometry of the problem. Recall that $\mathcal{S} \hookrightarrow J^2(\mathbb{C}, \mathbb{C}^2)$ refers to the subbundle of $J^2(\mathbb{C}, \mathbb{C}^2)$ defined by the system $\{F[u] = 0, u - f[v] = 0\}$. It sits above the double fibration given by

$$\begin{array}{ccccc} & & \mathcal{S} & & \\ & \pi_1 \swarrow & & \searrow \pi_2 & \\ J^2(\mathbb{C}, \mathbb{C}) \hookleftarrow \mathcal{F} & & & & \mathcal{H} \hookrightarrow J^3(\mathbb{C}, \mathbb{C}), \end{array}$$

where the subbundles $\mathcal{F}$ and $\mathcal{H}$ are defined by $F[u] = 0$ and $H[u] = 0$, respectively. The pull-back of $\mu[V]H[V] = \frac{d}{dx}G[V]$ with respect to the right projection π_2 (which changes U to V', U' to V'', and leaves V invariant) yields $e^{6V}F[U] = \frac{d}{dx}(e^{6V}I[U])$, where $I[U] = U' + \frac{1}{3}U^3 - 3U^2$ (now $\frac{d}{dx}$ is the total derivative operator induced by system $\mathcal{S}$). In other words, the function $e^{6V}I[U]$ is a first integral for the ODE system $\mathcal{S}$. The only way to get rid of V is to make it equal to zero. □

The double fibration in the previous example is usually called a Bäcklund transformation. In general, the data given by Eqs. (10), (11), and (13) always define a Bäcklund transformation in which π_2 changes U to $f(x, V, V')$, U' to $\frac{d}{dx}f(x, V, V')$ etc., and leaves V invariant.

Example. As another example, consider the class of differential equations arising from reduction of the nonlinear diffusion equation under scaling invariance of the form

$$2\left(K(u)u'\right)' + xu' = 0. \tag{24}$$

One can show that ODE (24) has a point symmetry if and only if

$$K(u) = a(u+b)^c, \tag{25}$$

where $\{a, b, c\}$ are arbitrary constants, or the limiting case

$$K(u) = ae^{bu}. \tag{26}$$

In particular, Lie's symmetry reduction method can only reduce the order of (24) when $K(u)$ is of the form (25) or (26). In Bluman and Reid (1988), "new symmetries" were found for the class of ODEs (24) when

$$K(u) = \frac{1}{au^2 + bu + c}\exp\left(\lambda\int\frac{1}{au^2 + bu + c}du\right), \tag{27}$$

where $\{a, b, c, \lambda\}$ are arbitrary constants. For arbitrary $K(u)$, one can easily check that the Bäcklund transformation $u = v'$ transforms the ODE (24) to the conserved form ODE

$$\frac{d}{dx}\left(2K(v')v'' + xv' - v\right) = 0. \tag{28}$$

One can show that a corresponding related second-order ODE

$$2K(v')v'' + xv' - v = 0 \tag{29}$$

has a point symmetry if and only if $K(u)$ is of the form (27). As an example, when $K(u) = \frac{1}{u^2+1}$, ODE (29) has point symmetry with infinitesimal generator $X = \left(x + vv'\right)\frac{\partial}{\partial v}$. In Bluman and Reid (1988), it was shown how this led to the general solution of ODE (29) and, in turn, to the general solution of ODE (24). Indeed, the mapping $u = v'$ when restricted to (29) yields the contact transformation $u = v', u' = v'' = \frac{v - xv'}{2K(v')}$. □

Example. If $F[u] = 0$ is the Riccati equation $u' + a_2(x)u^2 + a_1(x)u + a_0(x) = 0$, then the computation yields the well-known Bäcklund transformation $f[V] = \frac{\psi'(V)}{\psi(V)a_2(x)}V'$, where $\psi(V)$ is an arbitrary function of V. □

4. Classical and non-classical functions in the sense of Umemura

An ordinary differential field K is a field endowed with a derivation. If K is a subfield of a differential field L and it is stable under the derivation of L, then K, endowed with the same derivation, is a differential field, and it is said to be a differential subfield of L. If Σ is a set of elements of L, the extension of K generated by Σ, denoted by $K\langle\Sigma\rangle$, is the smallest differential subfield of L containing Σ and K. The following definition is due to Umemura (1998).

Definition 1. An ordinary differential field extension K of the field of rational functions $\mathbb{C}(x)$ is said to be *classical* if there exist $y_1, \ldots, y_N$ such that the following hold.

(i) The field K is a differential subfield of the field $\mathbb{C}(x)\langle y_1, \ldots, y_N\rangle$.
(ii) For each i, one of the following conditions is satisfied:
  (a) y_i is a solution of a homogeneous linear ODE with coefficients in $\mathbb{C}(x)\langle y_1, \ldots, y_{i-1}\rangle$ [Picard-Vessiot extension].

(b) y_i is the composite function $\psi(a_1, \ldots, a_n)$ for a certain Abelian function ψ[3] and certain $a_1, \ldots, a_n \in \mathbb{C}(x)\langle y_1, \ldots, y_{i-1}\rangle$ [Abelian extension].

An element of a classical extension is called a *classical function*.

The primitive of a classical function is also classical (since $y'(x) = a(x)$ implies the second-order homogeneous linear ODE $\left(y'/a\right)' = 0$ for y). It is also true, although not clear from the definition, that any solution of $f^n + a_1 f^{n-1} + \cdots + a_0 = 0$ is a classical function provided that the a_i are classical. Such solutions are said to be algebraic (over the appropriate field). However, transcendental classical functions are solutions of homogeneous linear ODEs with classical coefficients (or Abelian functions which arise rarely in practice). The Airy function solution of $v'' - xv = 0$ is a transcendental classical function. Other examples include Bessel functions, hypergeometric functions, and the error function.

The solution $u(x) = \frac{9}{1+\mathcal{W}(c_1 e^{-27x})}$ of ODE (1) is a transcendental classical function. The integrated equation with $c = 0$ simplifies to $\left(v'' + \frac{v'^3}{3} - 3v'^2\right) = 0$, which can be mapped to the homogeneous linear ODE $w''(t) + 3w'(t) = 0$ under the invertible bi-rational transformation $w(t) = x - v/2, t = v(x)$. Consequently, $v(x)$ is a transcendental classical function, and so is $u(x)$, the primitive of $v(x)$. The general solution of (1) is not necessarily a classical function, i.e., there is no invertible transformation $u = f(x, v, v') \in K(v, v')$ mapping ODE (1) to (a second-order) homogeneous linear ODE (here K is a classical extension of $\mathbb{C}(x)$).

An example of a non-classical function is the general solution of the first Painlevé equation $v'' - 6v^2 - x = 0$. If this Painlevé equation happens to be an integrated ODE for some second-order ODE, then this latter equation has no solutions which are classical functions (otherwise $v(x)$ satisfies the first-order ODE $u(x) - f(x, v(x), v'(x)) = 0$, which contradicts the irreducibility of the Painlevé equation).

As an example, it is now shown how the method presented here yields all classical solutions of the second Painlevé equation $u'' - 2u^3 - xu - \alpha = 0$. We obtain $(\mu[V], f_\alpha[V], \alpha) = \left(\psi(V), \alpha \frac{\psi'(V)}{\psi(V)} V', \pm\frac{1}{2}\right)$. Set $\psi(V) = 1/V^2$. The mapping $f_{-\frac{1}{2}}[v]$ reduces to $u = -\frac{1}{2}\frac{v'}{v}, u' = u^2 + \frac{1}{2}x - cv^2$, and the integrated ODE is the Airy equation when $c = 0$. This yields the well known Airy-like solution of the second Painlevé equation. Furthermore, since $f_{-\frac{1}{2}}[v]$ and $f_{+\frac{1}{2}}[v]$ map to the same Airy equation, the symmetry $s_1 : (u, \alpha) \to (-u, -\alpha)$ is constructed. Its invariant solution is simply $u(x) = 0$. On the other hand, leaving $u' - u^2 - \frac{1}{2}x = 0$ invariant, one obtains a second symmetry, $s_2 : (u, \alpha) \to (u + \frac{\alpha + \frac{1}{2}}{u' + u^2 + \frac{1}{2}x}, -1 - \alpha)$. It has been proven by the Japanese school (Umemura, 1998) that the reflections s_1 and s_2 (which generate a Weyl affine group of type $A_1^{(1)}$, where $u' - u^2 + \frac{1}{2}x = 0$ corresponds to a wall of a Weyl chamber of this affine group) generate together with the two above seed solutions ($u(x) = 0$ and the Airy-like solution) all classical solutions of the second Painlevé equation. Surprisingly, when $c \neq 0$, the integrated ODE is again Painlevé II with the non-zero constant c as a parameter. The contact transformation $u = -\frac{1}{2}\frac{v'}{v}, u' = u^2 + \frac{1}{2}x - cv^2$ is the folding transformation (Tsuda et al., 2005) mapping algebraic solutions to Airy-like solutions of Painlevé II.

5. Nonlinear oscillator equations

In this section, we consider the nonlinear oscillator equation

$$u'' + A(u)u' + B(u) = 0, \tag{30}$$

[3] Recall that an Abelian function is a meromorphic function on the complex torus (Abelian variety) $\mathbb{C}^p/\Gamma$ where $\Gamma \subset \mathbb{C}^p$ is a lattice. Abelian functions generalize the concept of elliptic functions (i.e., double-periodic meromorphic functions on the complex plane such as the Weierstrass function) for $p > 1$. Condition (b) simply means we may have to solve Weierstrass-like equations.

where $A(u), B(u) \in \mathbb{Q}(u)$; $\mathbb{Q}(u)$ is the field of rational functions in u with rational coefficients. (We are interested only in real ODEs. Moreover, it is sufficient to consider rational coefficients since real numbers are finitely represented in computer computation as *mantissa* $\times$ *base*^{exponent}, i.e., as rational numbers.)

Before starting our investigation, we make the remark that, in the generic case, the above ODE, just like the example in the introduction, has a one-dimensional point symmetry group: the translations $x \to x + \varepsilon$. In terms of the corresponding invariants $t = u$ and $w = u'$, Eq. (30) reduces to

$$\left(w'(t) + A(t)\right) w(t) + B(t) = 0, \tag{31}$$

which is an Abel equation of the second kind (Polyanin and Zaitsev, 1995).

5.1. The general case

Since the nonlinear oscillator equation (30) has arbitrary functions $A(u)$ and $B(u)$, we restrict the mapping (11) to be of the form $U = V' = V_1$. Moreover, we restrict the integrating factor (12) to be a function of V. In this case, the determining equation (17) splits into the two ODEs

$$\begin{cases} V_1^3\mu'''(V) - V_1^2 A(V_1)\mu''(V) - \left(B(V_1) - V_1 B'(V_1)\right)\mu'(V) = 0, \\ 3V_1\mu''(V) - \left(2A(V_1) + V_1 A'(V_1)\right)\mu'(V) + B''(V_1)\mu(V) = 0. \end{cases} \tag{32}$$

System (32) has the family of solutions $\mu(V) = c_0 e^{c_1 V}$ provided that the functions $A(V_1)$ and $B(V_1)$ are related by the restriction

$$A(V_1) = \frac{c_1^2 V_1^3 + V_1 B'(V_1) - B(V_1)}{c_1 V_1^2} \quad \text{or, equivalently,}$$

$$B(V_1) = V_1 \left(\int (c_1 A(V_1) - c_1 V_1) dV_1 + c_2 \right), \tag{33}$$

where c_0, c_1, and c_2 are arbitrary constants ($c_0 c_1 \neq 0$). It is easy to see that $A(V_1)$ is a polynomial function, i.e., $A(V_1) \in \mathbb{Q}[V_1]$ if and only if $B(V_1) \in \mathbb{Q}[V_1]$ and $B(0) = 0$. Also, $\deg(B(V_1)) = \deg(A(V_1)) + 2$. We set c_0 to 1. In terms of $B(v_1)$, the corresponding integrated equation is

$$G[v] = e^{c_1 v} \left(v'' + \frac{B(v_1)}{c_1 v_1} \right) = 0.$$

The mapping $u = v'$ restricted to $G[v] = c$ is given by

$$u = v', \qquad u' = v'' = -\frac{B(u)}{c_1 u} + c e^{-c_1 v}. \tag{34}$$

The mapping (34) is invertible if and only if c is not zero. In the generic case, the ODE $G[v] = c$ has only a one-dimensional point symmetry group (the translations $x \to x + \varepsilon$), and this is not enough to integrate it. However, when $c = 0$, provided that $A(u)$ and $B(u)$ are subject to the restriction (33), we have for the nonlinear oscillator equation (30) the one-parameter family of solutions $x + \int \left(\frac{c_1 u}{B(u)}\right) du + c_3 = 0$ corresponding to the general solution of the separable equation $u' + \frac{B(u)}{c_1 u} = 0$.

Now, consider specific examples.

5.2. Nonlinear oscillator with $A(u) = a_4 u^4 + a_3 u^3 + a_2 u^2 + a_1 u + a_0$

One obtains six cases where $B(u)$ is a polynomial in u and $B(0) = 0$.

5.2.1. Case I: $A(u) = u^4 + 4b_5 u^3 + 3b_4 u^2 + (1 + 2b_3)u + b_2$, $B(u) = (1/5)u^6 + \sum_1^5 b_i u^i$ and $\mu(V) = e^V$

Here, ODE (30) takes the form

$$u'' + \left(u^4 + 4b_5u^3 + 3b_4u^2 + (1+2b_3)u + b_2\right)u' + (1/5)u^6 + \sum_1^5 b_iu^i = 0. \tag{35}$$

The point symmetry group of ODE (35) is one dimensional. The integrated equation is

$$G[v] := e^v\left(v'' + \left(\frac{B(v')}{v'}\right)\right) = c, \tag{36}$$

which induces the mapping

$$u = v_1, \quad u' = -\left(\frac{B(u)}{u}\right) + ce^{-v},$$

which is invertible if and only if $c \neq 0$.

Proposition 2. *When $c \neq 0$, the point symmetry group of the integrated differential equation (36) is given by the translations $x \to x + \varepsilon$ induced by the original equation (35). Moreover, there is no further restrictions on the parameters which enlarges the admitted point symmetry group.*

Proof. The unique characteristic set of the Lie defining equations with respect to the evident ranking on the differential variables ξ, η, a_i and b_i is $\{\xi_x = 0, \xi_v, \eta = 0\}$. Here, $\xi = \xi(x, v)$ and $\eta = \eta(x, v)$ represent the infinitesimals. □

Furthermore, one can only obtain for the second-order ODE (35) a one-parameter family of solutions and not its general solution. So, setting c to zero, the corresponding one-parameter family of solutions is the general solution of the separable first-order differential equation

$$u' + \left(\frac{B(u)}{u}\right) = 0. \tag{37}$$

Suppose that $\frac{B(u)}{u} = \prod_{i=1}^5 (u - r_i)$, where $r_1 < r_2 < \cdots < r_5$ are real roots. Then, the general solution of the first-order ODE (37) is given by

$$x + c_1 + \sum_{i=1}^5 \frac{\ln(u(x) - r_i)}{\left[\frac{d}{du}\left(\frac{B(u)}{u}\right)\right]_{u=r_i}} = 0,$$

where c_1 is an arbitrary constant.

If $\frac{B(u)}{u} = (u - r)^5$, where r is real, then one has the algebraic general solution given by

$$x + c_1 - \frac{1}{4(u(x) - r)^5} = 0,$$

where c_1 is an arbitrary constant.

5.2.2. Case II: $A(u) = u^3 + 3b_4u^2 + (1 + 2b_3)u + b_2$, $B(u) = (1/4)u^5 + \sum_1^4 b_iu^i$ and $\mu(V) = e^V$

Here, ODE (30) takes the form

$$u'' + \left(u^3 + 3b_4u^2 + (1+2b_3)u + b_2\right)u' + (1/4)u^5 + \sum_1^4 b_iu^i = 0. \tag{38}$$

The point symmetry group of ODE (35) is one dimensional. The integrated ODE is

$$G[v] := e^v\left(v'' + \left(\frac{B(v')}{v'}\right)\right) = c.$$

This induces the mapping $u = v', u' = -\left(\frac{B(u)}{u}\right) + ce^{-v}$. The determinant of the corresponding Jacobian is ce^{-v}, which vanishes if and only if $c = 0$. There is a similar proposition as in the previous case. When $c = 0$, ODE (38) has a one-parameter family of solutions given by the general solution of the separable first-order ODE

$$u' + \left(\frac{B(u)}{u}\right) = 0. \tag{39}$$

Suppose that $\frac{B(u)}{u} = \prod_{i=1}^{4}(u - r_i)$, where $r_1 < r_2 < r_3 < r_4$ are real roots. Then the general solution of the first-order ODE (39) is given by

$$x + c_1 + \sum_{i=1}^{4} \frac{\ln(u(x) - r_i)}{\left[\frac{d}{du}\left(\frac{B(u)}{u}\right)\right]_{u=r_i}} = 0,$$

where c_1 is an arbitrary constant.

5.2.3. Case III: $A(u) = u^2 + (1 + 2b_3)u + b_2$, $B(u) = (1/3)u^4 + \sum_1^3 b_i u^i$ and $\mu(V) = e^V$

The corresponding ODE is

$$u'' + \left(u^2 + (1 + 2b_3)u + b_2\right) u' + (1/3)u^4 + \sum_1^3 b_i u^i = 0. \tag{40}$$

Again, the admitted point symmetry group is one dimensional, and no further restrictions on the parameters will enlarge it. One obtains a one-parameter family of solutions given by the general solution of the separable first-order ODE

$$u' + \left(\frac{B(u)}{u}\right) = 0. \tag{41}$$

Suppose that $\frac{B(u)}{u} = (u - r_1)(u - r_2)(u - r_3)$, where $r_1 < r_2 < r_3$ are real roots. Then, the general solution of the first-order ODE (41) is given by

$$x + c_1 + \sum_{i=1}^{3} \frac{\ln(u(x) - r_i)}{\left[\frac{d}{du}\left(\frac{B(u)}{u}\right)\right]_{u=r_i}} = 0,$$

where c_1 is an arbitrary constant.

5.2.4. Case IV: $A(u) = \frac{2a_0^2 + b_2^2}{a_0 b_2} u + a_0$, $B(u) = u^3 + b_2 u + b_1 u$ and $\mu(V) = e^{3\frac{b_2}{a_0} V}$

The corresponding ODE (30) is

$$u'' + \left(\frac{2a_0^2 + b_2^2}{a_0 b_2} u + a_0\right) u' + u^3 + b_2 u + b_1 u = 0, \tag{42}$$

which again has a one-dimensional point symmetry group. The resulting integrated ODE is

$$G[v] := e^{\frac{b_2}{a_0} v} \left(v_2 + \frac{a_0}{b_2}\left(\frac{B(v')}{v'}\right)\right) = c.$$

This induces the mapping

$$u = v', \quad u' = -\frac{a_0}{b_2}\left(\frac{B(u)}{u}\right) + c e^{-\frac{b_2}{a_0} v}.$$

The determinant of the corresponding Jacobian is

$$\frac{b_2}{a_0} c e^{-\frac{b_2}{a_0} v},$$

which vanishes if and only if $c = 0$ (since $a_0 b_2$ cannot vanish in this case). Setting $c = 0$, ODE (42) has a one-parameter family of solutions given by the general solution of the separable first-order ODE

$$u' + \frac{a_0}{b_2}\left(u^2 + b_2 u + b_1\right) = 0,$$

which can be represented as

$$u(x) = -\frac{1}{2}\left(b_2 - \tanh\left(\frac{1}{2}\frac{a_0\sqrt{\Delta}}{b_2}(x+c_1)\right)\sqrt{\Delta}\right) \qquad \text{when} \quad \Delta > 0,$$
$$u(x) = -\frac{1}{2}\left(b_2 + \tan\left(\frac{1}{2}\frac{a_0\sqrt{-\Delta}}{b_2}(x+c_1)\right)\sqrt{-\Delta}\right) \qquad \text{when} \quad \Delta < 0,$$
$$u(x) = -\frac{1}{2}\left(b_2 - \frac{2}{a_0(x+c_1)}\right) \qquad \text{when} \quad \Delta = 0,$$

where $\Delta = b_2^2 - 4b_1$ is the discriminant of $\frac{B(u)}{u} = u^2 + b_2 u + b_1$.

Remark 2. In Case IV, one can set $c \neq 0$ (here, the mapping is invertible, and hence one can obtain the general solution). One finds four cases where, with further restrictions on the parameters, the integrated ODE is solvable. However, in all these restricted cases, Maple was able to find the general solution directly from the original ODE.

5.2.5. *Case V: $A(u) = a_1 u$ and $B(u) = b_3 u^3 + b_1 u$*

Here, ODE (30) becomes

$$u'' + a_1 u u' + b_3 u^3 + b_1 u = 0.$$

This case is solvable by Lie's method (by Maple). Indeed, here, the reduced Abel equation

$$(w'(t) + a_1 t)w(t) + b_3 t^3 + b_1 t = 0$$

has symmetry $\frac{\left(2b_3 w^2 + (t^2 a_1 b_3 + a_1 b_1)w + b_3^2 t^4 + 2b_1 b_3 t^2 + b_1^2\right)}{w}\partial_w$.

5.2.6. *Case VI: $A(u) = a_1 u$ and $B(u) = b_1 u + b_0$*

ODE (30) becomes

$$u'' + a_1 u u' + (b_1 u + b_0) = 0. \tag{43}$$

For this case, excluding the trivial subcases, we were unable to find any solution. One can prove that the solutions of ODE (43) are non-classical functions.

5.3. *Nonlinear oscillator with $B(u) = b_4 u^4 + b_3 u^3 + b_2 u^2 + b_1 u + b_0$*

One can also consider the nonlinear oscillator equations (where $B(0) = b_0 \neq 0$)

$$u'' + A(u)u' + b_4 u^4 + b_3 u^3 + b_2 u^2 + b_1 u + b_0 = 0.$$

According to the constraint (33), $A(u)$ is a rational (not a polynomial) function of u. We consider only two cases here.

5.3.1. *Case 1: $A(u) = \frac{1}{u^2} + a_0 + a_1 u + a_2 u^2$, $B(u) = u^4 + \frac{1}{2}\frac{(3a_1 a_2 - 9)}{a_2^2}u^3 + \frac{3a_0}{a_2}u^2 + b_1 u - \frac{3}{a_2}$ and $\mu(V) = c_0 e^{\frac{3}{a_2}V}$*

Suppose that $B(u) = \prod_{i=1}^{4}(u - r_i)$, where $r_1 < r_2 < r_3 < r_4$ are real roots; then one obtains the one-parameter family of solutions

$$x + c_1 + 6a_2 \sum_{i=1}^{4} \frac{r_i\,(\ln(u(x) - r_i))}{[B'(u)]_{u=r_i}} = 0,$$

where c_1 is an arbitrary constant.

5.3.2. Case 2: $A(u) = \frac{1}{u^2} + u$, $B(u) = b_1 u - 1$ and $\mu(V) = b_2 + b_3 e^V$

Here, ODE (30) takes the form

$$u'' + \left(\frac{1}{u^2} + u\right) u' + b_1 u - 1 = 0. \tag{44}$$

The corresponding separable first-order ODE is

$$u' - \frac{1}{u} + b_1 = 0.$$

This yields the one-parameter family of solutions of ODE (44) given by

$$u(x) = \frac{1}{b_1}\left(\mathcal{W}\left(e^{-(b_1^2 x + c_1)}\right) + 1\right),$$

where c_1 is an arbitrary constant and $\mathcal{W}(x)$ is the Lambert W function.

5.4. Solutions for Abel equations

The first-order ODE $u'(x) + \frac{B(u)}{c_1 u} = 0$ yields an explicit solution for the Abel equation

$$\left(w'(t) + A(t)\right) w(t) + B(t) = 0,$$

where $A(t) = \frac{c_1^2 t^3 + tB'(t) - B(t)}{c_1 t^2}$, since $t = u(x)$ and $w(t) = u'(x)$. For instance, for ODE (44), the corresponding first-order ODE is

$$u' - \frac{1}{u} + b_1 = 0. \tag{45}$$

Writing (45) in terms of $t = u(x)$ and $w(t) = u'(x)$ yields $w(t) = \frac{1}{t} - b_1$, a solution of the Abel equation

$$\left(w'(t) + \frac{1}{t^2} + t\right) w(t) + b_1 t - 1 = 0.$$

Similarly, $w(t) = -\frac{1}{3}t^3 + 3t^2$ is a solution of the Abel equation (2) in the introduction.

5.5. The integrating factor as a function of x

Of course one can restrict the integrating factor (12) to be a function of x instead of being a function of V [The mapping (11) is again $U = V'$]. For example, if

$$A(u) = B'(u) + c_0, \qquad B(u) = (1/5)u^5 + \sum_0^4 b_i u^i \text{ and } \mu = c_0 e^x, \tag{46}$$

then one obtains the one-parameter family of solutions $x + c_1 + \sum_{i=1}^{5} \frac{\ln(u(x) - r_i)}{\left[B'(u)\right]_{u=r_i}} = 0$ if $B(u) = \prod_{i=1}^{5}(u - r_i)$, where $r_1 < r_2 \cdots < r_5$.

In the previous cases of Section 5.2, where the integrating factor $\mu = \mu(V)$, one had $\deg(B(u)) = \deg(A(u)) + 2$. On the other hand, in (46), one has $\deg(B(u)) = \deg(A(u)) + 1$.

6. Concluding remarks

In this paper, we have introduced a method for finding particular solutions of ODEs. Such solutions are not obtainable through direct use of the Maple ODE solver dsolve (which depends on finding and using integrating factors, Lie point symmetries, and contact symmetries). Our method, presented in detail in Section 3, uses a combination of invertible and non-invertible mappings. Many new solutions were found for various nonlinear oscillator equations.

In a future paper (Bluman and Dridi, in preparation), we will present in more detail the theoretical background of the new method presented in this paper.

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