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Torsional post-buckling of thin-walled open section clamped beam supported on Winkler-Pasternak foundation

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ABSTRACT

The study presents the post-buckling behaviour of thin-walled beam of open section supported by Winkler-Pasternak foundation and it is subjected to an axial compressive load. Here, the assumptions are the strains to be small and elastic, in-plan cross-sectional deformations and the shear deformations to be negligible. The post-buckling paths are determined for clamped beam of I - cross-section which is constant. The point of bifurcation for clamped beam is calculated. It is found to be symmetric and stable for various values of Winkler-Pasternak foundation parameters and Warping parameter.

1. Introduction

The study of the post-buckling response of thin-walled prismatic beam of open section has many applications in civil, naval, rail, aircraft and automotive structures. Furthermore, several of the base-frame structures of rotating equipment consist of thin-walled beams of open section which are either partially or continuously supported by other structural members or concrete foundations. Under some critical loading conditions these beams undergo either coupled flexural-torsional or pure torsional buckling which poses critical design problems in industry. Practical problems involving beams supported on a grid-type support structure or different types of machine foundations come under the category of beams on Winkler-Pasternak foundation and this improved model is being adopted for getting more accurate dynamic characteristics of the supported beam in bending, torsion and coupled bending-torsion type of problems.

The problem of linear torsional buckling has been widely investigated and the results for such cases are presented [1,2]. Although, the classical linear buckling theories for elastic beams necessarily predict buckling at loads that remain constant as the buckling amplitude increases. YOUNG [3] was the first person who investigated the nonlinear behaviour of circular members in uniform torsion. The related problem of torsional stiffness of narrow rectangular sections under axial tension was examined by BUCKLEY [4]. The behaviour of thin-walled I and Z sections was investigated by CULLIMORE [5]. GHOBARAH and TSO [6,7] presented more accurate theory of non-linear non-uniform torsion of thin-walled beams of open section by using the principle of minimum potential energy and taking into account large torsional deformations under general loading and boundary conditions.

Bazant and Nimeiri [8], Epstein and Murray [9], Szymzak [10], Roberts and Azizian [11] and Wekezer [12,13] studied the nonlinear torsional behaviour of thin-walled beams in a great detail. The post buckling behaviour of thin-walled open cross-section compression members using the general nonlinear theory of elastic stability was studied by Grimaldi et al. [14]. However, in all these studies the effect of continuous elastic foundation was not considered. Kameswara Rao et al. [15,16] studied the effect of Winkler-type elastic foundation on the linear torsional stability of thin-walled beams of open section, but its effect on post-buckling behaviour was not studied. KAMESWARA RAO ET AL., [17] investigated the post-buckling behaviour of thin-walled beams of open section subjected to an axial compressive load and resting on a Winkler-type continuous elastic foundation.

Winkler model of elastic foundation, also known as one-parameter model, is the simplest and the widely used model which is based on the assumption that the respective displacement is proportional to the pressure at the contact surface all along the beam length and that the soil foundation is composed of closely spaced, independent and linear elastic springs. As this Winkler model suffers from the deficiency of assuming that no interaction takes place between these soil springs, it does not accurately predict the dynamic response of practical foundations such as pavements or machine foundations [18]. In order to overcome this deficiency, a two-parameter foundation model such as Winkler-Pasternak foundation model is developed by PASTERNAK [19] who introduced an additional shear layer in the Winkler model and thus including the effect of shear interactions between the springs.

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Nomenclature

A	Area of cross-section and length of the beam	P	Axial compressive load
L	Length of the beam	P_{cr}	Linear buckling load
G	Shear modulus	P_{cr}^*	Non-linear buckling load
E	Young's modulus	σ	P/A, axial compressive stress
$\varnothing$	Angle of twist	K^2	GC_SL^2/EC_w Warping parameter
$x(z)$	Normal function of angle of twist	Δ^2	$\sigma I_PL^2/EC_w$; load parameter
z	Distance along the length of beam	Δ_{cr}	Linear bucking load parameter
C_W	Warping constant	Δ_{cr}^*	Non-linear bucking load parameter
C_S	Torsion constant	F	$I_R - (I_{PC}/A)^2$, cross-sectional property
K	Warping parameter	I_R	Polar moment of inertia of beam, also $I_{PC} = 1/2I_P$
K_w	Winkler foundation modulus	δ	F/C_w
K_p	Pasternak foundation modulus	λ^2	K_wL^4/EC_w Winkler foundation parameter
		ξ^2	K_pL^2/EC_w Pasternak foundation parameter

SIMÃO [20] presented post-buckling bi-furcational analysis of thin-walled prismatic members in the context of the generalized beam theory. He presented a series of analytical models, based on the generalized beam theory (GBT), investigating the buckling and post-buckling behaviour of thin-walled prismatic cold-formed steel structural members under compression and/or bending. The effect of warping constraints on the buckling of thin-walled structures was studied by Pignataro et al. [21]. They have considered various warping constraints at the bar ends and the relevant buckling modes and loads are numerically evaluated. Camotim et al. [22] presented a state-of-the art report on the use of Generalized Beam Theory (GBT) to assess the buckling behaviour of plane and space thin-walled steel frames. The torsion of restrained thin-walled bars of open constant bi-symmetric cross-section was investigated by Kujawa [23] using Castiglione's first theorem. The exact solutions were simplified by expanding them in a power series. The effect of warping on the post-buckling behaviour of thin-walled structures was investigated by Rizzi and Varano [24]. They have used a direct one-dimensional model to describe the mechanical behaviour of thin-walled beams to analyse the initial post buckling of some sample framed structures. Kujava and Szymczak [25] studied the elastic stability of axially compressed bar related to the cross–section distortion using the principle of stationary total potential energy. The aim of present paper is to study in detail, the effect of continuous Winkler-Pasternak elastic foundation on the torsional post-buckling behaviour of clamped uniform thin-walled beam of open cross-section.

2. Mathematical formulation of the problem

The purpose of this work is to study theatrically the elastically torsional post buckling behaviour of statically indeterminate (or hyperstatic) beam of I-section supported on Winkler-Pasternak foundation as shown in Fig. 1(a). The beam under consideration is doubly symmetric thin-walled beam of constant cross-section undergoing pure torsion and subjected to an axial compressive load. The constant cross-section of I-section is shown in Fig. 1(b).

Assumptions in the present study are, neglect the effects of (i) shear deformations (ii) large and inelastic strains, and (iii) in-plane cross-sectional deformations.

Considering $\varnothing$ as the angle of twist undergone by the thin-walled open section beam, the torque developed in the beam T under non-linear torsional deformation is given by

$$T = GC_S\varnothing' - EC_W\varnothing''' + 2\,EF(\varnothing')^3 \tag{1}$$

where C_w is the warping constant $= I_fh^2/2$, C_S = shear constant= (1/3) $(2b_ft_f^3 + ht_w^3)$, the prime denotes differentiation with respect to z, E the Young's modulus, G the shear modulus, I_R the fourth moment of inertia about the shear centre, I_{PC} the half of the polar moment of inertia about the shear center and F is a constant depending on cross-sectional properties of the beam defined as $F = I_R - (I_{PC}/A)^2$.

Further, the reaction offered by the foundation (FR) is given by

$$FR = K_w\varnothing - K_P\varnothing'' \tag{2}$$

where K_w is the Winkler foundation modulus and K_P is the modulus of Pasternak foundation.

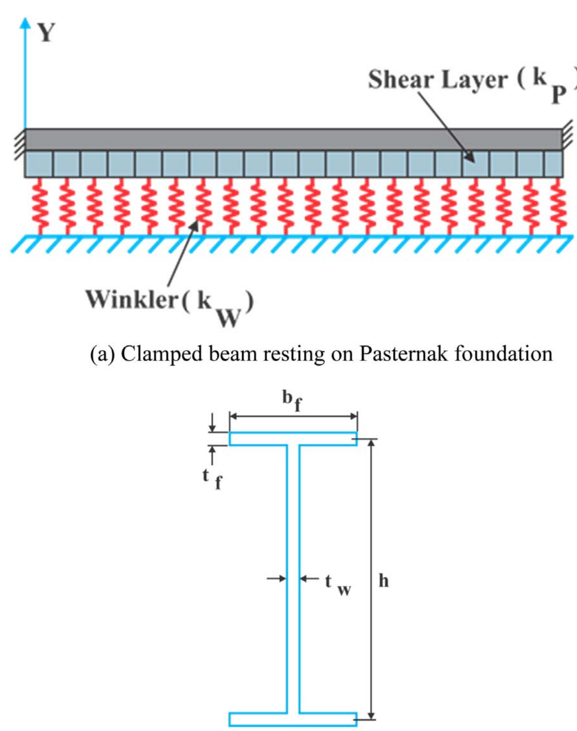
Considering a thin – walled doubly symmetric I-beam as shown in Fig. 1(a) and (b) having web and flange thickness t_f and t_w respectively, height between the centre lines of the flanges, h, flange width b_f, and flange and web thickness being assumed as small compared with height h, i.e., $t_f \ll h$ and $t_w \ll h$, the geometric properties in Eq. (1) can be evaluated as follows [6]

$$I_R = \frac{h^5t_w}{320} + \frac{b_fh^4t_f}{32} + \frac{b_f{}^5t_f}{160} + \frac{b_f{}^3h^2t_f}{48} \tag{3}$$

where

$$I_{PC} = \frac{1}{24}(h^3t_w + 2b_f{}^3t_f + 6b_fh^2t_f) \tag{4}$$

The total potential energy, consisting of the strain energy of deformation of the beam, the work done by the external axial



(a) Clamped beam resting on Pasternak foundation

(b) American steel thin-walled I-beam

Fig. 1. (a) Clamped beam resting on Pasternak foundation. (b) American steel thin-walled I-beam.

compressive load and the reaction offered by the continuous elastic foundation, is given by [10,14,19].

$$V = \frac{1}{2}\int_0^L [EC_W(\varnothing'')^2 + (GC_S - \sigma I_P + K_P)(\varnothing')^2 + EF(\varnothing')^4 + K_w(\varnothing)^2]dz = 0 \tag{5}$$

The fundamental differential equation resulting from the Euler condition of stationary potential energy given by Eq. (5) can be expressed as

$$EC_W\varnothing^{iv} - 6EF(\varnothing')^2\varnothing'' - (GC_S - \sigma I_P + K_P)\varnothing'' + K_w\varnothing = 0 \tag{6}$$

where $\sigma = P/A$ is the axial compressive stress acting on the beam due to load P. (7)

The general solution of Eq. (6) with the clamped boundary condition can be obtained by numerical methods using computer techniques. Here, the Galerkin's technique is used to obtain the approximate solution.

Eq. (6) can be re-written in non-dimensional form as

$$\varnothing^{iv} - 6\delta(\varnothing')^2\varnothing'' - (K^2 - \Delta^2 + \xi^2)\varnothing'' + 4\lambda^2\varnothing = 0 \tag{8}$$

Here, the parameter $\delta=\frac{F}{C_W}$. To solve Eq. (6) by Galerkin's method, the angle of twist, $\varnothing(Z)$ is assumed to be of the form

$$\varnothing(Z) = \beta x(Z) \tag{9}$$

where β is the torsional amplitude. The approximate function $\varnothing(Z)$ is assumed to satisfy the boundary conditions. On substituting Eq. (9) in (8), the error function, ϵ can be estimated as follows

$$\epsilon = \beta[x^{iv} - 6\beta^2\delta(x')^2x'' - (K^2 - \Delta^2 + \xi^2)x'' + 4\lambda^2 x] \tag{10}$$

In order to minimize the error ϵ, the Galerkin's integral is given by the following equation:

$$\int_0^1 \epsilon x(z)dZ = 0 \tag{11}$$

Substituting Eqs. (10) into Eq. (11), the expression for the torsional post-buckling load for a thin-walled beam of open section can be obtained as

$$\Delta_{cr}^{*2} = (K^2 + \xi^2) + \left(\frac{I_1}{I_3}\right) + 6\beta^2\delta\left(\frac{I_2}{I_3}\right) - 4\lambda^2\left(\frac{I_4}{I_3}\right) \tag{12}$$

where

$$I_1 = \int_0^1 x^{iv}xdz;\quad I_2 = \int_0^1 (x')^2x''xdz;\quad I_3 = \int_0^1 x''xdz;\quad I_4 = \int_0^1 x^2dz \tag{13}$$

The case considered here is a beam with clamped edge on both the ends. The boundary conditions associated with clamped edges are:

$\varnothing = 0;\ \ \varnothing' = 0$ at $Z = 0$,

where the non − dimensional beam length Z = (z/L) (14)

$$\varnothing = 0;\ \ \varnothing' = 0 \text{ at } Z = 1 \tag{15}$$

We assume the following function for x(Z) which satisfies the clamped boundary conditions given by Eqs. (14) and (15) as

$$x(Z) = \beta(1 - \cos 2\pi Z) \tag{16}$$

Evaluating the integrals I_1,I_2,I_3 and I_4 given in Eq. (13) utilising the function given in Eq. (16) and substituting the same in Eq. (12) we obtain the expression for the critical post-buckling load as

$$\Delta_{cr}^{*2} = K^2 + \xi^2 + 4\pi^2 + \frac{3\lambda^2}{\pi^2} + 6\pi^2\delta\beta^2 \tag{17}$$

The corresponding linear torsional buckling load for a clamped beam is given by

$$\Delta_{cr}^2 = K^2 + \xi^2 + 4\pi^2 + \frac{3\lambda^2}{\pi^2} \tag{18}$$

Therefore, the ratio of the nonlinear to linear buckling load can be expressed as

$$\frac{P^*}{P_{cr}} = \frac{\Delta_{cr}^{*2}}{\Delta_{cr}^2} = 1 + (6\pi^4\delta\beta^2)/[\pi^2(K^2 + \xi^2 + 4\pi^2) + 3\lambda^2] \tag{19}$$

In the absence of Pasternak elastic foundation, i.e., $\xi = 0$, Eq. (19) reduces to

$$\frac{P^*}{P_{cr}} = \frac{\Delta_{cr}^{*2}}{\Delta_{cr}^2} = 1 + 6\pi^4\delta\beta^2/[\pi^2(K^2 + 4\pi^2) + 3\lambda^2] \tag{20}$$

In the absence of an Winkler elastic foundation, i.e., $\lambda^2 = 0$, Eq. (20) reduces to

$$\frac{P^*}{P_{cr}} = \frac{\Delta_{cr}^{*2}}{\Delta_{cr}^2} = 1 + 6\pi^4\delta\beta^2/[\pi^2(K^2 + 4\pi^2)] \tag{21}$$

3. Results and conclusions

Consider a doubly symmetric I-beam (see Fig. 1(b)) for the analysis with the following dimensions (all dimensions are in mm):

Length of the beam $(L)=760$; Web thickness $(t_w)=2.13$;Flange thickness $(t_f)=3.11$; Flange width $(b_f)=31.55$; Depth of the beam $(d)=72.76$; Distance between center lines of flanges $= h=69.65$;.

The non-dimensional parameters K and δ are determined from the beam properties and obtained as $K = 3.106$ and $\delta = 1.1095$. The ratio of nonlinear buckling load to linear buckling load can be determined from Eq. (20). Arithmetical values of the ratio of nonlinear buckling load to linear buckling load (P^*/P_{cr}) against values of torsional amplitude (β) for different values of Winkler stiffness parameter (λ) and Pasternak stiffness parameter (ξ) are obtained using Eq. (20).

The values of the ratio of nonlinear buckling to linear buckling loads (P^*/P_{cr}) against torsional amplitude (β) for various values of Winkler stiffness parameter $(\lambda = 0,5,10,15,20,25\ \&\ 30)$ by keeping Pasternak stiffness parameter constant $(\xi = 0)$ are obtained and results are plotted in Fig. 2. Similarly, the values of the ratio of nonlinear buckling to linear buckling loads (P^*/P_{cr}) against torsional amplitude (β) for various values of Winkler stiffness parameter $(\lambda = 0,5,10,15,20,25\ \&\ 30)$ by keeping Pasternak stiffness parameter constant $(\xi = 25,50,75\ \&\ 100)$ are obtained and the resulting plots are given in Figs. 3–6. The selected values of Winkler and Pasternak stiffness parameters $(\lambda\ \&\ \xi)$, provide a wide range of foundations characteristics varying from no torsional stiffness to high torsional stiffness values. It is observed from the Fig. 2, that the ratio of nonlinear buckling load to linear torsional buckling load (P^*/P_{cr}) increases with increasing values of the torsional amplitude

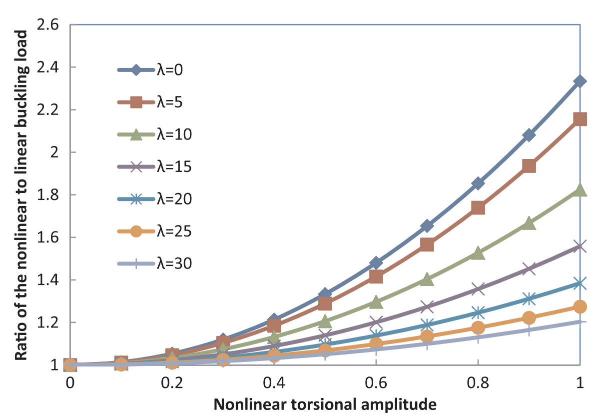


Fig. 2. The effect of elastic foundation (for various values of Winkler stiffness parameter, λ, with Pasternak stiffness parameter, ξ=0) on torsional post-buckling behaviour of clamped thin-walled beam of open section.

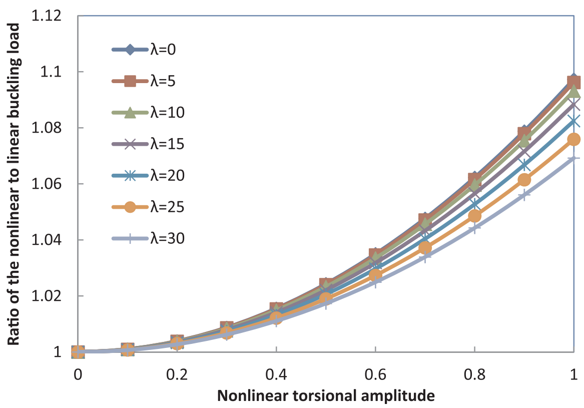


Fig. 3. The effect of elastic foundation (for various values of Winkler stiffness parameter, λ, with Pasternak stiffness parameter, ξ=25) on torsional post-buckling behaviour of clamped thin-walled beam.

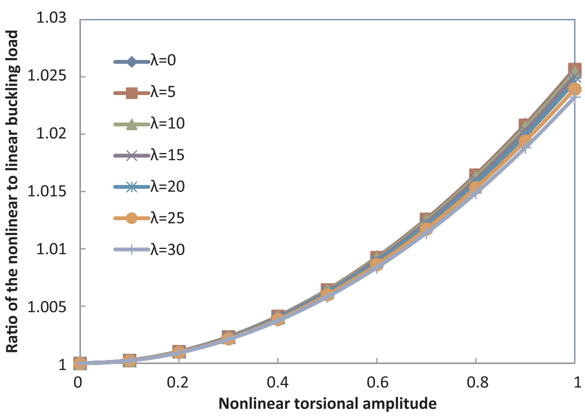


Fig. 4. The effect of elastic foundation (for various values of Winkler stiffness parameter, λ, with Pasternak stiffness parameter, ξ=50) on torsional post-buckling behaviour of clamped thin-walled beam.

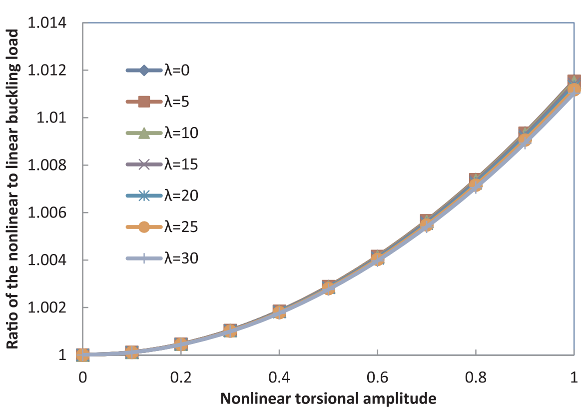


Fig. 5. The effect of elastic foundation (for various values of Winkler stiffness parameter, λ, with Pasternak stiffness parameter, ξ=75) on torsional post-buckling behaviour of clamped thin-walled beam.

(β) for a given values of Winkler stiffness parameter ($\lambda = 0,5,10,15,20,25 \,\&\, 30$) by keeping Pasternak stiffness parameter, $\xi = 0$. The equilibrium configurations in the torsional post-buckling region exist only for axial compressive loads in excess of the critical load of small deflection theory. It is also observed that for lower values of , the nonlinear buckling load increases rapidly as β increases. Also, as λ increases, the nonlinear buckling load decreases as β increases as shown in Figs. 2–6. Also, as λ increases, the curves become flatter indicating that the influence of β on nonlinear buckling load becomes gradually less significant.

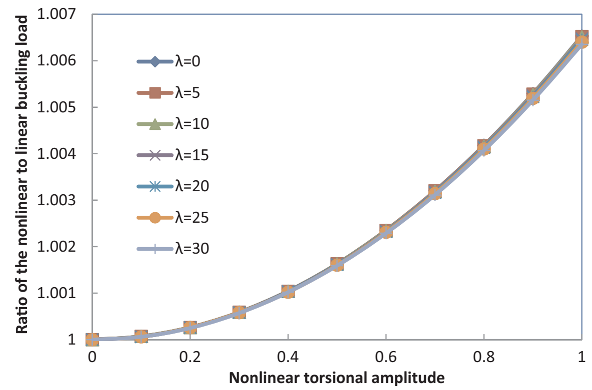


Fig. 6. The effect of elastic foundation (for various values of Winkler stiffness parameter, λ, with Pasternak stiffness parameter, ξ=100) on torsional post-buckling behaviour of clamped thin-walled beam.

From the results presented in Figs. 2–6, we can easily observed that the ratio of the nonlinear to linear torsional buckling load (P^*/P_{cr}) increases consistently with increasing values of torsional amplitude (β). The values of ratio of the nonlinear to linear torsional buckling load (P^*/P_{cr}) increases with increasing values of the torsional amplitude (β) for various values of Winkler stiffness parameter, λ, as Pasternak stiffness parameter, ξ increases from 0 to 100. Also, for various values ofλ, the influence of β on nonlinear buckling load becomes gradually less significant as ξ increases.

The percentage variation of the ratio of nonlinear to linear buckling load (P^*/P_{cr}) as β varies from 0.1 to 1, with Winkler stiffness parameter, λ, for various values of Pasternak stiffness parameter, ($\xi = 0,25,50,75 \,\&\, 100$) is computed and shown in Fig. 7. The percentage increase of the ratio of nonlinear to linear buckling load (P^*/P_{cr}) is more for $\xi = 0$ and decreases with increase inλ. Also, the percentage variation is less significant as ξ increases.

The values of the ratio of nonlinear to linear buckling loads (P^*/P_{cr}) against torsional amplitude (β) for various values of Pasternak stiffness parameter in the range ($\xi = 0,5,10,15,20,25 \,\&\, 30$) are obtained by keeping Winkler stiffness parameter constant ($\lambda = 0$) are obtained and results are plotted in Fig. 8. Similarly, the values of the ratio of nonlinear to linear buckling loads (P^*/P_{cr}) against torsional amplitude (β) for various values of Pasternak stiffness parameter ($\xi = 0,5,10,15,20,25 \,\&\, 30$) by keeping Winkler stiffness parameter constant ($\lambda = 25,50,75 \,\&\, 100$) are obtained and the resulting plots are shown in Figs. 8–12.

It can be easily observed from the Fig. 8, that the ratio of nonlinear to linear torsional buckling load (P^*/P_{cr}) increases with increasing values of the torsional amplitude (β) for a given values set of values of Pasternak stiffness parameter ($\xi = 0,5,10,15,20,25 \,\&\, 30$) by keeping

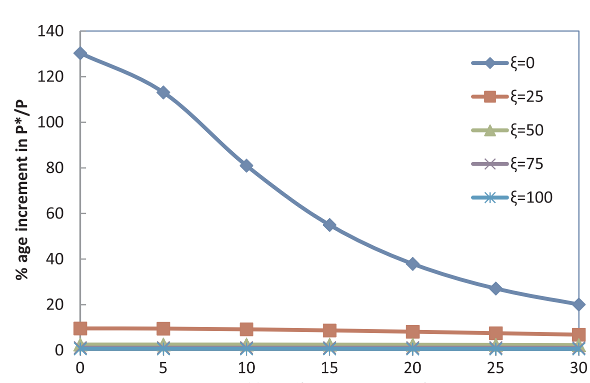


Fig. 7. The percentage variation of the ratio of nonlinear to linear buckling load (P^*/P_{cr}) as β varies from 0.1 to 1, with Winkler stiffness parameter, λ, for different values of Pasternak stiffness parameter, ξ=0,25,50,75 & 100).

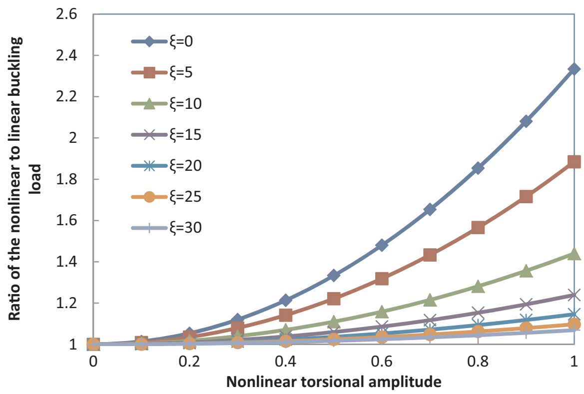


Fig. 8. The effect of elastic foundation (for various values of Pasternak stiffness parameter, ξ, with Winkler stiffness parameter, λ=0) on torsional post-buckling behaviour of clamped thin-walled beam.

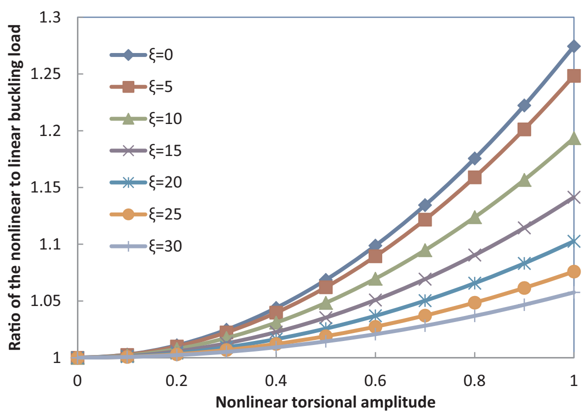


Fig. 9. The effect of elastic foundation (for various values of Pasternak stiffness parameter, ξ, with Winkler stiffness parameter, λ=25) on torsional post-buckling behaviour of clamped thin-walled beam.

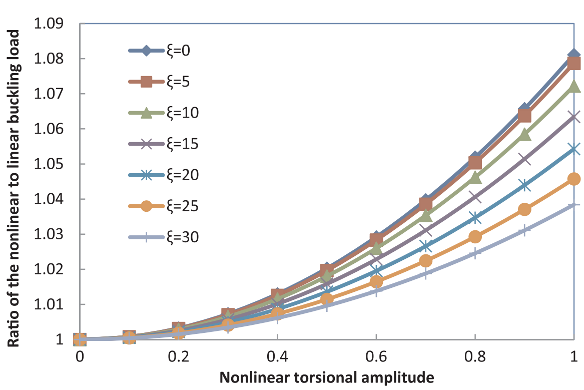


Fig. 10. The effect of elastic foundation (for various values of Pasternak stiffness parameter, ξ, with Winkler stiffness parameter, λ=50) on torsional post-buckling behaviour of clamped thin-walled beam.

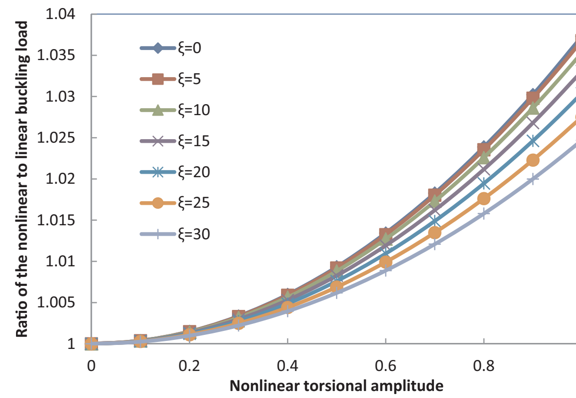


Fig. 11. The effect of elastic foundation (for various values of Pasternak stiffness parameter, ξ, with Winkler stiffness parameter, λ=75) on torsional post-buckling behaviour of clamped thin-walled beam.

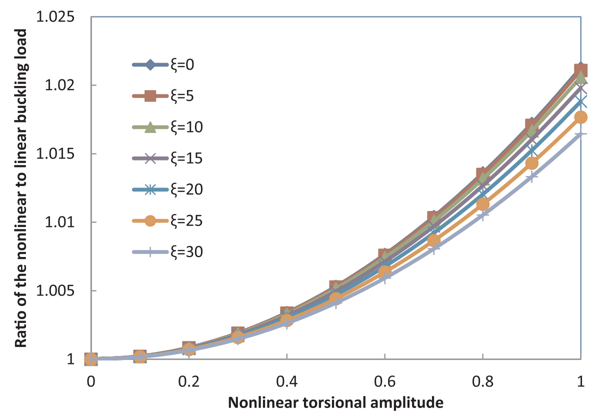


Fig. 12. The effect of elastic foundation (for various values of Pasternak stiffness parameter, ξ, with Winkler stiffness parameter, λ=100) on torsional post-buckling behaviour of clamped thin-walled beam.

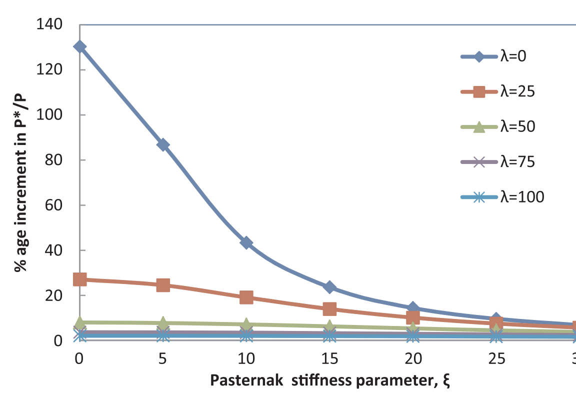


Fig. 13. The percentage variation of the ratio of nonlinear to linear buckling load (P^*/P_{cr}) as β varies from 0.1 to 1, with Pasternak stiffness parameter, ξ, for different values of Winkler stiffness parameter, λ=0,25,50,75 & 100).

Winkler stiffness parameter, $\lambda = 0$. It is also observed that for lower values of , the nonlinear buckling load increase rapidly as β increases. Also, as ξ increases, the nonlinear buckling load decreases as β increases as shown in Figs. 8–12. Also, as ξ increases, the curves become flatter indicating that the influence of β on nonlinear buckling load becomes gradually decreases.

From the Figs. 8–12 presented, it can be also observed that the ratio of the nonlinear to linear torsional buckling load (P^*/P_{cr}) increases with increasing values of torsional amplitude (β). The values of the ratio of nonlinear to linear torsional buckling load (P^*/P_{cr}) can be seen to be increasing with increasing values of the torsional amplitude (β) for various values of ξ, as λ increases from 0 to 100. Also, for different values of ξ, the influence of β on nonlinear buckling load becomes gradually less significant as λ increases.

The percentage variation of the ratio of nonlinear to linear buckling load (P^*/P_{cr}) as β varies from 0.1 to 1, with Pasternak stiffness parameter, ξ, for various values of Winkler stiffness parameter, (λ=0,25,50,75 & 100) is computed and shown in Fig. 13. The percentage increase of the ratio of nonlinear to linear buckling load (P^*/P_{cr}) is more for $\lambda = 0$ and decreases with increase inξ. Also, the percentage variation is less significant as λ increases.

The variation of the nonlinear buckling load with increasing values of torsional amplitude for different values of $\lambda$is more convergence (as ξ increases from 0 to 100, shown in Figs. 2–6) than the variation of the nonlinear buckling load with increasing values of torsional amplitude for different values of ξ as λ increases from 0 to 100 (as shown in Figs. 8–12). It is observed from the Eq. (19) that as δ increases the nonlinear buckling load P^* increases for constant values of β, K, ξ & λ. It is also observed that the effect of increase in values of Warping parameter, K, Pasternak stiffness parameter ξ, and/or Winkler stiffness parameter λ is to decrease the nonlinear buckling load, P^* considerably. It is noticed that the rate of change in the nonlinear buckling load, P^* is gradually reduces due to increase in β & λ as ξ increases and for any constant values of K & δ (Refer Figs. 2–6). It is also observed that the rate of change in the nonlinear buckling load P^* is more significant due to increase in β & ξ as λ increases and for any constant values of K & δ (Refer Figs. 8–12).

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