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Two models of three-dimensional thin interphases with variable conductivity and their fulfillment of the reciprocal theorem

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ABSTRACT

Interphases appear in heterogeneous media in a variety of forms. Often the treatment of a thin interphase as a separate phase in a multiphase solid is not convenient in analytical or numerical solutions of those systems. Thus, approximate models of a thin interphase that make possible to obtain a solution for the fields in the media adjacent to it, without the need of determining the fields within the interphase itself, become a necessity in many cases. The question then arises whether a global property which was present in the original heterogeneous medium will continue to prevail after an approximate representation of the thin interphase has been introduced in the system. A global property, known to have important consequences on the behavior of the heterogeneous solid, is the "reciprocity" relation between a pair of two different solutions, as stated by the reciprocal theorem. Since the formulation of an approximate model for the thin interphase involves several assumptions, the fulfillment of the reciprocal theorem in the original system does not necessarily imply its fulfillment in the transformed system in which an approximate model of the thin interphase has been intoduced. The preservation of the reciprocity relation by the approximate model, if proved, would be considered to be an important consistency quality of the model. In this paper we consider steady thermal conduction phenomena, and generalize the two approximate models of a thin interphase by Bövik (1994), Benveniste (2006), Benveniste and Berdichevsky (2010) to the case of thin interphases with a variable conductivity. The fulfillment of the reciprocity property in the presence of those models, which was not discussed in the above papers, is investigated here in the context of their presently developed generalized version, and it is proved that both models fulfill the reciprocal theorem.

1. Introduction

Interphases appear in heterogenous media in a variety of forms. They are encountered in adhesive joints and are also present in the form of coatings in composites. The treatment of a thin interphase as a separate phase is often not convenient in analytical or numerical solutions of those systems. For example, graded interphases with a complex variation of their moduli through their thickness may render an analytical solution to be inaccessible in the heterogeneous solid. On the other hand, finite element solutions, when applied to systems involving thin interphases, necessitate elongated elements which are known to be undesirable in those methods. Thus, approximate models of a thin interphase which make possible to obtain a solution for the fields in the media adjacent to it without the need of determining the fields within the interphase itself, become a necessity in many cases.

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A literature review on the modeling of thin interphases can be found, for example, in Benveniste (2006) and in a recent comprehensive study on the subject by Gu and He (2011). In those works an "interface model" of a thin interphase was formulated, in which the interface comes into direct contact with the media which are adjacent to the interphase, and that is characterized by appropriate "interface conditions" for the fields. The developed interface model of the interphase in the above two studies is of $O(t)$ accuracy where t is the thickness of the thin interphase. In Benveniste and Berdichevsky (2010) an additional model was proposed in which the geometry of the interphase is left intact, and is characterized by conditions pertaining to the adjacent media which are evaluated at both sides of the interphase. Those conditions do not involve the fields within the interphase, but yet they depend on its material properties and on those of the adjacent media as well, and make possible to determine the fields in the adjacent media without the need of solving the fields within the interphase. Both $O(t)$ and $O(t^2)$ accuracy versions of that second model were developed in the latter paper, and their performance was tested in the setting of anti-plane and in-plane elasticity problems by comparing their predictions with analytical solutions carried out in the exact three-phase inclusion-coating-matrix configuration.

The present study is a continuation of the Benveniste (2006), and Benveniste and Berdichevsky (2010) papers. It is formulated in the setting of steady thermal conduction, and its contribution is twofold: (a) the two $O(t)$ models in those papers, in which a constant conductivity of the interphase was assumed, are generalized here to the case of non-constant interphase conductivities, (b) it is shown that in systems containing a thin interphase, in which the so-called "Reciprocal Theorem" is fulfilled, this theorem will continue to be valid after the interphase has been replaced by either one of the above approximate models. An important consistency property of those models will be thus established.

Consider steady conduction in multiphase media with a symmetric conductivity tensor k_{ij} and in which the temperature and normal heat flux fields are continuous at the interphase boundaries. The reciprocal theorem in this system is a surface integral relation between a pair of two different solutions for the temperature and normal heat flux fields evaluated on the external surface of the solid. Specifically, let a heterogeneous body of volume V be surrounded by a surface Γ on which it is subjected to two different sets of normal heat fluxes denoted by q_n and q_n'. In that exact setting of steady heat conduction, and under no heat sources, the reciprocal theorem states that

$$\int_\Gamma q_n \phi' d\Gamma = \int_\Gamma q_n' \phi d\Gamma, \tag{1.1}$$

where ϕ' and ϕ are the temperature fields induced by the normal heat fluxes q_n' and q_n, respectively. It is well known that the fulfillment of (1.1) has important consequences on the behavior of a heterogeneous system. For example, it implies that its exact effective conductivity tensor is symmetric, and also produces symmetric conductivity matrices in finite-element solutions, reducing thus considerably the amount of the involved computational effort. Therefore, in systems which contain thin interphases in which the reciprocal theorem holds, the question of whether the same property will continue to be valid after the replacement of the interphase by a certain approximate model is an important one. Since the representation of thin interphases by approximate models involves several assumptions, there is no guarantee that a global property which was present in the original system, as the one embodied by the fulfillment of the reciprocal theorem, will continue to prevail after an approximate model of the interphase has been introduced in it, and needs to be verified in each particular case. For example, Chen (2001) showed that the "weakly conducting interface model" (Kapitza-type), and its dual "superconducting interface model" (used in cases when they would be applicable) fulfill the reciprocal theorem. The models in Benveniste (2006) and Benveniste and Berdichevsky (2010), which are generalized here to the case of an interphase with a variable conductivity, have the advantage of being applicable to a wide spectrum of the interphase conductivity, and incorporate the weakly conducting and the superconducting interface models as special cases. In the present study we prove that that in systems containing thin interphases where the reciprocal theorem is fulfilled, this theorem continues to be true after the interphase has been replaced by either one of the above models.

The analysis in the paper is carried out for the case of a three-dimensional and arbitrarily curved thin interphase. In Section 2 of the paper we briefly review first the elements of "parallel orthogonal curvilinear coordinate system" suitable in treating such a general geometry. Then, in a solid with variable conductivity we derive the representations for the normal derivatives of the temperature and normal heat flux on a surface in terms of surface differential forms. Those results are instrumental in the formulation of both models which is carried out in Section 3. In Section 4, the validity of the reciprocal theorem in the presence of both models is proved. Since the derived models are of $O(t)$ accuracy, the fulfillment of the reciprocal theorem is sought within $O(t)$ accuracy only. As shown in that section, a certain transformation on the temperature and normal heat flux fields turns out to play a major role in the achievement of the proofs. The paper concludes with an Appendix which makes contact with a recent two-dimensional study of Sussmann et al. (2011) which is concerned with the incorporation of approximate interphase models in finite-element formulations.

2. Preliminaries

2.1. The description of a constant thickness interphase by a parallel orthogonal curvilinear coordinate system

Consider an interphase of constant thickness separating two media. Let the interphase be denoted as Medium 0 and the adjacent media as Medium 1 and Medium 2. The mid-surface of the interphase is denoted by S_0, and its inner and outer

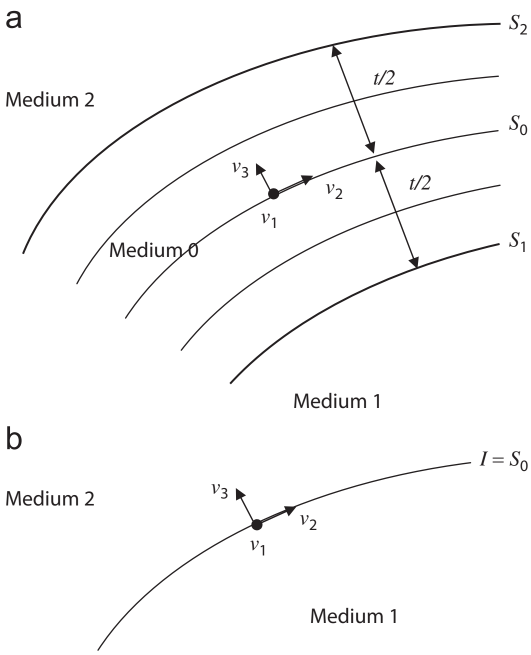

Fig. 1. (A) The geometry of a thin interphase described by a curvilinear orthogonal parallel coordinate system. (B) The configuration in which the interface I comes into direct contact with Medium I and Medium II.

surfaces by S_1 and S_2, see Fig. 1a. The mid-surface S_0 is parametrically described by

$$x_{S_0}=x_{S_0}(v_1,v_2),\quad y_{S_0}=y_{S_0}(v_1,v_2),\quad z_{S_0}=z_{S_0}(v_1,v_2), \tag{2.1}$$

where x_{S_0},y_{S_0},z_{S_0} are the coordinates in a Cartesian system possessing the unit vectors $\mathbf{i},\mathbf{j},\mathbf{k}$, and $v_1=\text{constant}$, $v_2=\text{constant}$ are the lines of curvature of the surface S_0 which are orthogonal to each other. Let $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$ be the unit vectors tangent to the lines of curvature of S_0. This interphase will be geometrically characterized by means of a parallel orthogonal curvilinear coordinate system described below. The purpose of this subsection is to review certain geometrical connections in this system which will be used in the main part of the paper. The analysis here is based on Appendix 3 of Van Bladel (2006), and Appendix A of Benveniste (2006).

Consider a system consists of surfaces parallel to the surface S_0 and construct a developable surface generated by the normals to the lines of curvature $v_1=\text{constant}$, $v_2=\text{constant}$ of S_0. In virtue of this construction, those normals intersect all parallel surfaces along their lines of curvature. The parallel surfaces together with the developable surface constitute a "parallel orthogonal curvilinear coordinate system". What in particular characterizes such a system is the fact that it can be built up on any smooth surface of an arbitrary shape.

Next, consider a parallel surface S located at a normal distance v_3 from the surface S_0. If the position vector of a point P at the intersection of two lines of curvatures on the surface S_0 is defined by

$$\mathbf{r}_P=x_{S_0}(v_1,v_2)\mathbf{i}+y_{S_0}(v_1,v_2)\mathbf{j}+z_{S_0}(v_1,v_2)\mathbf{k}, \tag{2.2}$$

then the position vector at a corresponding point Q at the surface S, located along the normal raised at P is

$$\mathbf{r}_Q=\mathbf{r}_P+v_3\hat{\mathbf{u}}_3, \tag{2.3}$$

where $\hat{\mathbf{u}}_3$ is the unit vector normal to S_0, chosen to point away from the center of curvature of that surface, and v_3 is the linear coordinate along $\hat{\mathbf{u}}_3$, as measured from S_0. Thus, the parametric representation of the surface S is given by

$$\begin{aligned} x_S&=x_{S_0}(v_1,v_2)+(\hat{\mathbf{u}}_3\cdot\mathbf{i})v_3,\\ y_S&=y_{S_0}(v_1,v_2)+(\hat{\mathbf{u}}_3\cdot\mathbf{j})v_3,\\ z_S&=z_{S_0}(v_1,v_2)+(\hat{\mathbf{u}}_3\cdot\mathbf{k})v_3. \end{aligned} \tag{2.4}$$

Next we define the metric coefficients $h_r^{(S_0)}$, $h_r^{(S)}$ with $r=1,2$ affiliated to the lines of curvature of the surfaces S_0 and S, respectively. Those are:

$$\begin{aligned} h_r^{(S_0)}&=\left\{\left(\frac{\partial x_{S_0}}{\partial v_r}\right)^2+\left(\frac{\partial y_{S_0}}{\partial v_r}\right)^2+\left(\frac{\partial z_{S_0}}{\partial v_r}\right)^2\right\}^{1/2}=\left\{\left(\frac{\partial \mathbf{r}_P}{\partial v_r}\right)\cdot\left(\frac{\partial \mathbf{r}_P}{\partial v_r}\right)\right\}^{1/2},\\ h_r^{(S)}&=\left\{\left(\frac{\partial x_S}{\partial v_r}\right)^2+\left(\frac{\partial y_S}{\partial v_r}\right)^2+\left(\frac{\partial z_S}{\partial v_r}\right)^2\right\}^{1/2}=\left\{\left(\frac{\partial \mathbf{r}_Q}{\partial v_r}\right)\cdot\left(\frac{\partial \mathbf{r}_Q}{\partial v_r}\right)\right\}^{1/2},\quad r=1,2 \end{aligned} \tag{2.5}$$

while the metric coefficient along the linear coordinate ν_3 normal to the parallel surfaces are $h_3^{(S_0)}=h_3^{(S)}=1$. The metric coefficients along the lines of curvature of the parallel surfaces are related to the incremental arc-lengths $ds_r^{(S_0)}$, $ds_r^{(S)}$ along those curves by

$$ds_r^{(S_0)}=h_r^{(S_0)}d\nu_r,\quad ds_r^{(S)}=h_r^{(S)}d\nu_r,\quad r=1,2 \tag{2.6}$$

In view of this definition and the representation in (2.2) it follows that the quantities $h_r^{(S_0)}$ (with $r=1,2$) obey

$$h_r^{(S_0)}=\mathrm{Lim}_{\Delta\nu_r\to 0}\frac{\Delta s_r^{(S_0)}}{\Delta\nu_r}=\mathrm{Lim}_{\Delta\nu_r\to 0}\frac{\Delta\mathbf{r}_P\cdot\hat{\mathbf{u}}_r}{\Delta\nu_r}=\frac{\partial\mathbf{r}_P}{\partial\nu_r}\cdot\hat{\mathbf{u}}_r, \tag{2.7}$$

where $\Delta\mathbf{r}_P$ is an incremental vector lying on the surface S_0. Invoking now the connections (see, for example, Eq. (A.2) in Benveniste (2006))

$$\frac{\partial\hat{\mathbf{u}}_3}{\partial\nu_1}=\frac{\partial h_1^{(S_0)}}{\partial\nu_3}\hat{\mathbf{u}}_1,\quad \frac{\partial\hat{\mathbf{u}}_3}{\partial\nu_2}=\frac{\partial h_2^{(S_0)}}{\partial\nu_3}\hat{\mathbf{u}}_2, \tag{2.8}$$

makes possible to prove from (2.5) to (2.8) the following result

$$h_1^{(S)}=h_1^{(S_0)}\left(1-\frac{\nu_3}{R_1^{(S_0)}}\right),\quad h_2^{(S)}=h_2^{(S_0)}\left(1-\frac{\nu_3}{R_2^{(S_0)}}\right), \tag{2.9}$$

where $R_1^{(S_0)}$ and $R_2^{(S_0)}$ are the principal radii of curvature of S_0 which are defined as

$$\frac{1}{R_1^{(S_0)}}=-\frac{1}{h_1^{(S_0)}}\frac{\partial h_1^{(S_0)}}{\partial\nu_3},\quad \frac{1}{R_2^{(S_0)}}=-\frac{1}{h_2^{(S_0)}}\frac{\partial h_2^{(S_0)}}{\partial\nu_3}, \tag{2.10}$$

and result as negative quantities when the unit normal $\hat{\mathbf{u}}_3$ points away from the center of curvature, as in the present formulation. Using similar forms for the principal radii of curvature $R_1^{(S)}$ and $R_2^{(S)}$ of the surface S provides the following connections

$$R_1^{(S)}=R_1^{(S_0)}-\nu_3,\quad R_2^{(S)}=R_2^{(S_0)}-\nu_3. \tag{2.11}$$

We conclude this section by mentioning that an infinitesimal element of the surface S will be related to an infinitesimal element of the surface S_0 by

$$dS=h_1^{(S)}h_2^{(S)}d\nu_1 d\nu_2=\left(1-\frac{\nu_3}{R_1^{(S_0)}}\right)\left(1-\frac{\nu_3}{R_2^{(S_0)}}\right)h_1^{(S_0)}h_2^{(S_0)}d\nu_1 d\nu_2=\left(1-\frac{\nu_3}{R_1^{(S_0)}}\right)\left(1-\frac{\nu_3}{R_2^{(S_0)}}\right)dS_0. \tag{2.12}$$

The relations (2.9), (2.11) and (2.12) will play an important part in the analysis of Sections 3 and 4.

2.2. *The description of the normal derivatives of the temperature and normal heat flux along a surface with a variable conductivity*

Thermal conduction in isotropic media is governed by

$$\mathbf{q}=k\mathbf{H}, \tag{2.13}$$

where $\mathbf{q}$ is the heat flux vector, k is the isotropic conductivity which is assumed to be variable, $\mathbf{H}$ is the heat intensity vector defined by

$$\mathbf{H}=-\mathbf{grad}\,\phi, \tag{2.14}$$

and ϕ denotes the temperature. Under steady state conditions, the heat flux vector obeys the balance law

$$\mathrm{div}\,\mathbf{q}=0 \tag{2.15}$$

Consider now a surface S in a solid whose isotropic conductivity varies in space, and construct on it a parallel and orthogonal curvilinear coordinate system. The following representation can be shown to exist for the normal derivatives of the temperature and the normal heat flux along the surface S:

$$\frac{\partial\phi}{\partial\nu_3}=L(\phi)+Q(q_3),\quad \frac{\partial q_3}{\partial\nu_3}=R(\phi)+P(q_3), \tag{2.16}$$

where L,Q,R,P are surface differential forms involving surface derivatives on S. The explicit expressions derived below for those differential forms constitute a generalization of the results presented in Bövik (1994), and Benveniste (2006), to the case in which the conductivity k varies along the surface S.

From (2.13) and (2.14) one has

$$q_1\hat{\mathbf{u}}_1+q_2\hat{\mathbf{u}}_2+q_3\hat{\mathbf{u}}_3=-k(\nu_1,\nu_2)\mathbf{grad}\phi=-k(\nu_1,\nu_2)\left\{\left(\frac{1}{h_1^{(S)}}\right)\left(\frac{\partial\phi}{\partial\nu_1}\right)\hat{\mathbf{u}}_1+\left(\frac{1}{h_2^{(S)}}\right)\left(\frac{\partial\phi}{\partial\nu_2}\right)\hat{\mathbf{u}}_2+\left(\frac{\partial\phi}{\partial\nu_3}\right)\right\}\hat{\mathbf{u}}_3 \tag{2.17}$$

so that L,Q are immediately identified as

$$L(\phi)=0, \quad Q(q_3)=-\frac{q_3}{k(v_1,v_2)}, \tag{2.18}$$

where the variation of k on the surface S is denoted by $k(v_1,v_2)$.

On the other hand, in order to derive R and P of (2.16) it is first noted that

$$\text{div}\,\mathbf{q}=\text{div}_S\mathbf{q}+\frac{\partial q_3}{\partial v_3}=0, \tag{2.19}$$

in which the surface divergence $\text{div}_S\mathbf{q}$ of the heat flux is given by

$$\text{div}_S\mathbf{q}=\left(\frac{1}{h_1^{(S)}h_2^{(S)}}\right)\left(\frac{\partial}{\partial v_1}(h_2^{(S)}q_1)+\frac{\partial}{\partial v_2}(h_1^{(S)}q_2)\right)-\left(\frac{1}{R_1^{(S)}}+\frac{1}{R_2^{(S)}}\right)q_3. \tag{2.20}$$

Moreover, from (2.17) there is

$$q_1\hat{\mathbf{u}}_1+q_2\hat{\mathbf{u}}_2=-k(v_1,v_2)\mathbf{grad}_S\phi=-k(v_1,v_2)\left\{\left(\frac{1}{h_1^{(S)}}\right)\left(\frac{\partial\phi}{\partial v_1}\right)\hat{\mathbf{u}}_1+\left(\frac{1}{h_2^{(S)}}\right)\left(\frac{\partial\phi}{\partial v_2}\right)\hat{\mathbf{u}}_2\right\}, \tag{2.21}$$

where the surface gradient of ϕ is denoted by $\mathbf{grad}_S\phi$. Thus, from (2.19–2.20) it follows that

$$\frac{\partial q_3}{\partial v_3}=\text{div}_S(k(v_1,v_2)\mathbf{grad}_S\phi)+\left(\frac{1}{R_1^{(S)}}+\frac{1}{R_2^{(S)}}\right)q_3, \tag{2.22}$$

where

$$\begin{aligned}\text{div}_S(k(v_1,v_2)\mathbf{grad}_S\phi)&=\left(\frac{1}{h_1^{(S)}h_2^{(S)}}\right)\frac{\partial}{\partial v_1}\left\{k(v_1,v_2)\left(\frac{h_2^{(S)}}{h_1^{(S)}}\frac{\partial\phi}{\partial v_1}\right)\right\}+\left(\frac{1}{h_1^{(S)}h_2^{(S)}}\right)\frac{\partial}{\partial v_2}\left\{k(v_1,v_2)\left(\frac{h_1^{(S)}}{h_2^{(S)}}\frac{\partial\phi}{\partial v_2}\right)\right\}\\&=k(v_1,v_2)\,\Delta_S\phi+\mathbf{grad}_S\phi\cdot\mathbf{grad}_Sk(v_1,v_2),\end{aligned} \tag{2.23}$$

with the surface Laplacian $\Delta_S\phi$ being defined by

$$\Delta_S\phi=\frac{1}{h_1^{(S)}h_2^{(S)}}\left\{\frac{\partial}{\partial v_1}\left(\frac{h_2^{(S)}}{h_1^{(S)}}\frac{\partial\phi}{\partial v_1}\right)+\frac{\partial}{\partial v_2}\left(\frac{h_1^{(S)}}{h_2^{(S)}}\frac{\partial\phi}{\partial v_2}\right)\right\}. \tag{2.24}$$

Finally, from (2.22) the explicit expressions for R and P result as

$$R=\text{div}_S(k(v_1,v_2)\mathbf{grad}_S\phi), \quad P=\left(\frac{1}{R_1^{(S)}}+\frac{1}{R_2^{(S)}}\right)q_3. \tag{2.25}$$

3. Two models for a three-dimensional constant thickness thin interphase with a variable conductivity

In this section we formulate the generalization of the models in Benveniste (2006) and Benveniste and Berdichevsky (2010) to the case in which interphase has a variable conductivity. The model in which the interphase geometry is left intact, but is nevertheless characterized *only* by the fields belonging to the adjacent media that are evaluated at the surfaces of the interphase, will be called Model I; the one in which the interphase is replaced by an interface separating Medium 1 and 2 will be called Model II. Both models will be of $O(t)$ accuracy where t is the thickness of the interphase.

3.1. Model I

Consider a constant thickness thin interphase denoted as Medium 0 separating two media denoted by Medium 1 and Medium 2. Let the mid-surface of the interphase be denoted by S_0, and its inner and outer surfaces by S_1 and S_2, see Fig. 1a. The derivation starts by expressing the temperature at the mid-surface S_0 of the interphase in terms of Taylor expansions about lower surface S_1 and the upper surface S_2. This provides

$$(\phi^{(0)})_{v_3=0}=(\phi^{(0)})_{v_3=-t/2}+\left(\frac{t}{2}\right)\left(\frac{\partial\phi^{(0)}}{\partial v^3}\right)_{v_3=-t/2}+O(t^2), \tag{3.1}$$

$$(\phi^{(0)})_{v_3=0}=(\phi^{(0)})_{v_3=t/2}-\left(\frac{t}{2}\right)\left(\frac{\partial\phi^{(0)}}{\partial v^3}\right)_{v_3=t/2}+O(t^2). \tag{3.2}$$

Subtracting (3.2) from (3.1), yields

$$(\phi^{(0)})_{\nu_3=t/2}-(\phi^{(0)})_{\nu_3=-t/2}=\left(\frac{t}{2}\right)\left\{\left(\frac{\partial\phi^{(0)}}{\partial\nu^3}\right)_{\nu_3=t/2}+\left(\frac{\partial\phi^{(0)}}{\partial\nu^3}\right)_{\nu_3=-t/2}\right\}+O(t^2), \tag{3.3}$$

and using the representation for the normal derivatives of the temperature stated in (2.16) gives

$$\begin{aligned}\left(\phi^{(0)}\right)_{\nu_3=t/2}-\left(\phi^{(0)}\right)_{\nu_3=-t/2}&=(t/2)\Big[\left\{L^{(0)}(\phi^{(0)})\right\}_{\nu_3=t/2}+\left\{Q^{(0)}(q_3^{(0)})\right\}_{\nu_3=t/2}\\&\quad+\left\{L^{(0)}(\phi^{(0)})\right\}_{\nu_3=-t/2}+\left\{Q^{(0)}(q_3^{(0)})\right\}_{\nu_3=-t/2}\Big)\Big]+O(t^2)\,,\end{aligned} \tag{3.4}$$

where the superscript $(\ )^{(0)}$ appears now on the surface differential operators in order to indicate that they relate to the case in which the normal derivative in (2.16) was performed in Medium 0. The next step is to use the continuity conditions on the temperature and normal heat flux on the surfaces S_1 and S_2, which transforms (3.4) into

$$\begin{aligned}\left(\phi^{(2)}\right)_{\nu_3=t/2}-\left(\phi^{(1)}\right)_{\nu_3=-t/2}&=(t/2)\Big(\left\{L^{(0)}(\phi^{(2)})\right\}_{\nu_3=t/2}+\left\{Q^{(0)}(q_3^{(2)})\right\}_{\nu_3=t/2}\\&\quad+\left\{L^{(0)}(\phi^{(1)})\right\}_{\nu_3=-t/2}+\left\{Q^{(0)}(q_3^{(1)})\right\}_{\nu_3=-t/2}\Big)+O(t^2).\end{aligned} \tag{3.5}$$

An analogous development on the normal heat fluxes provides the following dual expression at S_1 and S_2

$$\begin{aligned}\left(q_3^{(2)}\right)_{\nu_3=t/2}-\left(q_3^{(1)}\right)_{\nu_3=-t/2}&=(t/2)\Big(\left\{R^{(0)}(\phi^{(2)})\right\}_{\nu_3=t/2}+\left\{P^{(0)}(q_3^{(2)})\right\}_{\nu_3=t/2}\\&\quad+\left\{R^{(0)}(\phi^{(1)})\right\}_{\nu_3=-t/2}+\left\{P^{(0)}(q_3^{(1)})\right\}_{\nu_3=-t/2}\Big)+O(t^2).\end{aligned} \tag{3.6}$$

The next step is to invoke the explicit expressions of the surface differential forms (2.18) and (2.25). This provides:

$$\left(\phi^{(2)}\right)_{\nu_3=t/2}-\left(\phi^{(1)}\right)_{\nu_3=-t/2}=-\left(\frac{t}{2}\right)\left\{\left(\frac{1}{k_0^{(S_2)}(\nu_1,\nu_2)}\right)\left(q_3^{(2)}\right)_{\nu_3=t/2}+\left(\frac{1}{k_0^{(S_1)}(\nu_1,\nu_2)}\right)\left(q_3^{(1)}\right)_{\nu_3=-t/2}\right\}+O(t^2), \tag{3.7}$$

$$\begin{aligned}\left(q_3^{(2)}\right)_{\nu_3=t/2}-\left(q_3^{(1)}\right)_{\nu_3=-t/2}&=\left(\frac{t}{2}\right)\left\{\left(\frac{1}{R_1^{(S_2)}}+\frac{1}{R_2^{(S_2)}}\right)\left(q_3^{(2)}\right)_{\nu_3=t/2}+\left(\frac{1}{R_1^{(S_1)}}+\frac{1}{R_2^{(S_1)}}\right)\left(q_3^{(1)}\right)_{\nu_3=-t/2}\right\}\\&\quad+\left(\frac{t}{2}\right)\left\{\mathrm{div}_{S_2}\left(k_0^{(S_2)}(\nu_1,\nu_2)\mathbf{grad}_{S_2}(\phi^{(2)})_{\nu_3=t/2}\right)+\mathrm{div}_{S_1}\left(k_0^{(S_1)}(\nu_1,\nu_2)\mathbf{grad}_{S_1}(\phi^{(1)})_{\nu_3=-t/2}\right)\right\}+O(t^2),\end{aligned} \tag{3.8}$$

where $k_0^{(S_1)}(\nu_1,\nu_2)$ and $k_0^{(S_2)}(\nu_1,\nu_2)$ denote the conductivity of the interphase along the surfaces S_1 and S_2 respectively, and where we have used the notation div_{S_1} and div_{S_2}in order to indicate that the operators are applied on those surfaces.

Eqs. (3.7) and (3.8) which govern Model I can be cast in alternative equivalent forms. Since we are dealing here with a model of $O(t)$ accuracy, and the right-hand sides of (3.7) and (3.8) contain already the thickness t, it first follows that in those expressions one is allowed to make the following replacements:

$$k_0^{(S_2)}(\nu_1,\nu_2)\Rightarrow k_0^{(S_0)}(\nu_1,\nu_2),\quad k_0^{(S_2)}(\nu_1,\nu_2)\Rightarrow k_0^{(S_0)}(\nu_1,\nu_2), \tag{3.9}$$

where $k_0^{(S_0)}(\nu_1,\nu_2)$ is the conductivity at the middle surface which for simplicity we will be indicated by $k_0^{(S_0)}(\nu_1,\nu_2)=k_0(\nu_1,\nu_2)$. It should be noted that such an approximation cannot be made for higher order models of interphases existing, for example, in Benveniste and Baum (2007), where the derivatives of $k_0(\nu_1,\nu_2,\nu_3)$ with respect to ν_3, and evaluated at $\nu_3=0$, will eventually enter in the model. An additional simplification of (3.7) and (3.8), made in the same spirit, consists in introducing the following replacements on the right-hand side of those equations:

$$\left(\frac{1}{R_1^{(S_2)}}+\frac{1}{R_2^{(S_2)}}\right)\Rightarrow\left(\frac{1}{R_1^{(S_0)}}+\frac{1}{R_2^{(S_0)}}\right),\quad\left(\frac{1}{R_1^{(S_1)}}+\frac{1}{R_2^{(S_1)}}\right)\Rightarrow\left(\frac{1}{R_1^{(S_0)}}+\frac{1}{R_2^{(S_0)}}\right), \tag{3.10}$$

$$\begin{aligned}&\mathrm{div}_{S_1}\left(k_0^{(S_1)}(\nu_1,\nu_2)\mathbf{grad}_{S_1}(\phi^{(1)})_{\nu_3=-t/2}\right)\Rightarrow\mathrm{div}_{S_0}\left(k_0(\nu_1,\nu_2)\mathbf{grad}_{S_0}(\phi^{(1)})_{\nu_3=-t/2}\right),\\&\mathrm{div}_{S_2}\left(k_0^{(S_1)}(\nu_1,\nu_2)\mathbf{grad}_{S_2}(\phi^{(2)})_{\nu_3=t/2}\right)\Rightarrow\mathrm{div}_{S_0}\left(k_0(\nu_1,\nu_2)\mathbf{grad}_{S_0}(\phi^{(2)})_{\nu_3=t/2}\right),\end{aligned} \tag{3.11}$$

where the definition in (2.23) and the result stated in (2.9) were invoked. The replacement (3.11) implies that although the temperature and the normal heat flux fields appearing in them are evaluated on the surfaces S_1 and S_2, their tangential derivatives can be performed with respect to coordinates affiliated to the mid-surface S_0. Use of (3.9)–(3.11) in (3.7) and (3.8) gives

$$\left(\phi^{(2)}\right)_{\nu_3=t/2}-\left(\phi^{(1)}\right)_{\nu_3=-t/2}=-\left(\frac{t}{2}\right)\left\{\left(\frac{1}{k_0(\nu_1,\nu_2)}\right)\left(q_3^{(2)}\right)_{\nu_3=t/2}+\left(\frac{1}{k_0(\nu_1,\nu_2)}\right)\left(q_3^{(1)}\right)_{\nu_3=-t/2}\right\}+O(t^2), \tag{3.12}$$

$$\left(q_3^{(2)}\right)_{v_3=t/2}-\left(q_3^{(1)}\right)_{v_3=-t/2}=\left(\frac{t}{2}\right)\left\{\left(\frac{1}{R_1^{(S_0)}}+\frac{1}{R_2^{(S_0)}}\right)\left(q_3^{(2)}\right)_{v_3=t/2}+\left(\frac{1}{R_1^{(S_0)}}+\frac{1}{R_2^{(S_0)}}\right)\left(q_3^{(1)}\right)_{v_3=-t/2}\right\}+$$
$$+\left(\frac{t}{2}\right)\left\{\mathrm{div}_{S_0}\left(k_0(v_1,v_2)\mathbf{grad}_{S_0}(\phi^{(2)})_{v_3=t/2}\right)+\mathrm{div}_{S_0}\left(k_0(v_1,v_2)\mathbf{grad}_{S_0}(\phi^{(1)})_{v_3=-t/2}\right)\right\}+O(t^2) \tag{3.13}$$

The decision of whether to choose (3.7), (3.8) or (3.12),(3.13) for Model II is a matter of convenience in the course of the implementation of that model. For example, it will be seen in Section 4 that (3.12) and (3.13) will be more convenient in the proof of the reciprocal theorem in the setting of Model I.

We finally make note of the fact that if in the paragraph above (3.1) and (3.2), instead of beginning the derivation with the mid-surface S_0, had we started with a surface S_d located at a normal distance d from the inner surface S_1, and had expressed the temperature and normal heat flux on S_d in terms of Taylor expansions about S_1 and S_2, we would have obtained the following expressions instead of (3.5) and (3.6)

$$\left(\phi^{(2)}\right)_{v_3=t/2}-\left(\phi^{(1)}\right)_{v_3=-t/2}=(t-d)\left(\left\{L^{(0)}(\phi^{(2)})\right\}_{v_3=t/2}+\left\{Q^{(0)}(q_3^{(2)})\right\}_{v_3=t/2}\right)+d\left(\left\{L^{(0)}(\phi^{(1)})\right\}_{v_3=-t/2}\right.$$
$$\left.+\left\{Q^{(0)}(q_3^{(1)})\right\}_{v_3=-t/2}\right)+O(t^2), \tag{3.14}$$

$$\left(q_3^{(2)}\right)_{v_3=t/2}-\left(q_3^{(1)}\right)_{v_3=-t/2}=(t-d)\left(\left\{R^{(0)}(\phi^{(2)})\right\}_{v_3=t/2}+\left\{P^{(0)}(q_3^{(2)})\right\}_{v_3=t/2}\right)+d\left(\left\{R^{(0)}(\phi^{(1)})\right\}_{v_3=-t/2}\right.$$
$$\left.+\left\{P^{(0)}(q_3^{(1)})\right\}_{v_3=-t/2}\right)+O(t^2) \tag{3.15}$$

In order to possess a symmetric structure to the model, in this study we will adopt the previous option resulting in (3.5) and (3.6) which have led to (3.12) and (3.13).

3.2. Model II

In Model II the interphase is replaced by an interface I onto which the adjacent media are made to come into direct contact, see Fig. 1b. As in Model I, in order to preserve a symmetric structure to the model we chose to position that interface at the location of the mid-surface S_0 of the interphase which has been removed.

The derivation starts by demanding that the conditions (3.5) and (3.6), which are valid at the locations $-t/2$ and $+t/2$ in the configuration of Fig. 1a are also valid at the same imaginary locations of Fig. 1b. With this requirement, the fields at the locations $v_3 \le -t/2$ and $v_3 \ge +t/2$ in both configurations of Fig. 1 become the same, justifying therefore the introduction of the two-phase configuration in Fig. 1b. In order to complete the construction of Model II, additional Taylor expansions need to be performed in Eqs. (3.5) and (3.6) in the setting of Fig. 1b. This procedure will be illustrated below for Eq. (3.5) with an analogous one being applicable to Eq. (3.6).

On the left hand side of (3.5), Taylor expansions for the temperature field in the configuration of Fig. 1b gives:

$$(\phi^{(2)})_{v_3=t/2}=(\phi^{(2)})_+ +(t/2)\left(\frac{\partial\phi^{(2)}}{\partial v_3}\right)_+ +O(t^2)=(\phi^{(2)})_+ +(t/2)\left(L^{(2)}(\phi^{(2)})_+ +Q^{(2)}(q_3^{(2)})_+\right)+O(t^2), \tag{3.16}$$

$$(\phi^{(1)})_{v_3=-t/2}=(\phi^{(1)})_- -(t/2)\left(\frac{\partial\varphi^{(1)}}{\partial v_3}\right)_- +O(t^2)=(\phi^{(1)})_- -(t/2)\left(L^{(1)}(\phi^{(1)})_- +Q^{(1)}(q_3^{(1)})_-\right)+O(t^2), \tag{3.17}$$

where the notation $(\ )_-$ and $(\ )_+$ indicates that the relevant quantities have been evaluated, respectively on the side of Medium 1 and Medium 2 at the introduced interface I in Fig. 1b. Note that the superscripts on the surface differential operators in (3.16) and (3.17) indicate their affiliation to the media adjacent to the interphase. On the other hand, since the right hand-side of (3.16) already contains the small parameter t, it is noted that for the development of an $O(t)$ theory it would be sufficient to have on that side expansions in the form:

$$\left\{L^{(0)}(\phi^{(2)})\right\}_{v_3=t/2}=\left\{L^{(0)}(\phi^{(2)})\right\}_+ +O(t),\quad \left\{L^{(0)}(\phi^{(1)})\right\}_{v_3=-t/2}=\left\{L^{(0)}(\phi^{(1)})\right\}_- +O(t), \tag{3.18}$$

with similar statements prevailing for the surface differential forms $Q^{(0)}$. Thus the first equation characterizing Model II becomes

$$\left(\phi^{(2)}\right)_+-\left(\phi^{(1)}\right)_-=(t/2)\left[\left\{L^{(0)}(\phi^{(2)})\right\}_+ +\left\{L^{(0)}(\phi^{(1)})\right\}_- +\left\{Q^{(0)}(q_3^{(2)})\right\}_+ +\left\{Q^{(0)}(q_3^{(1)})\right\}_-\right.$$
$$\left.-\left\{L^{(2)}(\phi^{(2)})\right\}_+ -\left\{L^{(1)}(\phi^{(1)})\right\}_- -\left\{Q^{(2)}(q_3^{(2)})\right\}_+ -\left\{Q^{(1)}(q_3^{(1)})\right\}_-\right]+O(t^2). \tag{3.19}$$

A similar derivation applied now to (3.6) gives

$$\left(q_3^{(2)}\right)_+-\left(q_3^{(1)}\right)_-=(t/2)\left[\left\{R^{(0)}(\phi^{(2)})\right\}_+ +\left\{R^{(0)}(\phi^{(1)})\right\}_- +\left\{P^{(0)}(q_3^{(2)})\right\}_+ +\left\{P^{(0)}(q_3^{(1)})\right\}_-\right.$$

$$-\left\{R^{(2)}(\phi^{(2)})\right\}_{+}-\left\{R^{(1)}(\phi^{(1)})\right\}_{-}-\left\{P^{(2)}(q_3^{(2)})\right\}_{+}-\left\{P^{(1)}(q_3^{(1)})\right\}_{-}\Bigg]+O(t^2). \tag{3.20}$$

Finally, substitution of (2.18) and (2.25) in (3.19) and (3.20), and for the case in which k_1 and k_2 are constant quantities, provides the following equations governing Model II:

$$\left(\phi^{(2)}\right)_{+}-\left(\phi^{(1)}\right)_{-}=\left(\frac{t}{2}\right)\left\{\left(\frac{1}{k_2}-\frac{1}{k_0(v_1,v_2)}\right)\left(q_3^{(2)}\right)_{+}+\left(\frac{1}{k_1}-\frac{1}{k_0(v_1,v_2)}\right)\left(q_3^{(1)}\right)_{-}\right\}+O(t^2), \tag{3.21}$$

$$\left(q_3^{(2)}\right)_{+}-\left(q_3^{(1)}\right)_{-}=\left(\frac{t}{2}\right)\Big\{\mathrm{div}_{S_I}\left(k_0(v_1,v_2)\mathbf{grad}_{S_I}(\phi^{(2)})_{+}\right)+\mathrm{div}_{S_I}\left(k_0(v_1,v_2)\mathbf{grad}_{S_I}(\phi^{(1)})_{-}\right)$$
$$-k_2\Delta_{S_I}(\phi^{(2)})_{+}-k_1\Delta_{S_I}(\phi^{(1)})_{-}\Big\}+O(t^2). \tag{3.22}$$

where it should be noted that the surface operators div_{S_I}, $\mathbf{grad}_{S_I}$ and Δ_{S_I} are to be evaluated on the interface I.

4. Proof of the fulfillment of the reciprocal theorem by Model I and Model II

In order to get a better perspective of the proof concerning the fulfillment of the reciprocal theorem by Model I and Model II which will be given in this section, we start with a brief review of this theorem in a heterogeneous medium, as present in the exact setting for the field equations. Without loss of generality, we choose that medium to be a three-phase solid consisting of an inhomogeneity, a coating and a matrix, of volumes V_1,V_0 and V_2, respectively, all surrounded by a surface Γ, see Fig. 2a. Let the surfaces separating the interphase from the inhomogeneity and the matrix by denoted by S_1 and S_2, respectively. We adopt the setting of steady heat conduction, with the temperature and normal heat flux being continuous at S_1 and S_2, and allow no heat sources in the body. Let the surface Γ be subjected to two different sets of normal heat fluxes denoted by q_n and q'_n. The reciprocal theorem states that

$$\int_{\Gamma} q_n\phi' d\Gamma=\int_{\Gamma} q'_n\phi d\Gamma, \tag{4.1}$$

where ϕ' and ϕ are the temperature fields induced by the heat fluxes q'_n and q_n, respectively. It is well known that the proof of this statement consists in showing that either side of (4.1) can be cast into a symmetric form in the unprimed and primed fields, and will thus be equal to each other. In that exact setting, the proof is achieved by starting with

$$J=\int_{V_1}(q_i^{(1)}\phi'^{(1)})_{,i}dV_1+\int_{V_0}(q_i^{(0)}\phi'^{(0)})_{,i}dV_0+\int_{V_2}(q_i^{(2)}\phi'^{(2)})_{,i}dV_2, \tag{4.2}$$

which, through the use of the divergence theorem, transforms into

$$J=\int_{S_1} q_i^{(1)}n_i^{(S_1)}\phi'^{(1)}dS_1+\int_{S_2} q_i^{(0)}n_i^{(S_2)}\phi'^{(0)}dS_2-\int_{S_1} q_i^{(0)}n_i^{(S_1)}\phi'^{(0)}dS_2+\int_{\Gamma} q_i^{(2)}n_i^{(\Gamma)}\phi'^{(2)}d\Gamma-\int_{S_2} q_i^{(2)}n_i^{(S_2)}\phi'^{(2)}dS_2, \tag{4.3}$$

where $n_i^{(S_1)}$,$n_i^{(S_2)}$, $n_i^{(\Gamma)}$ are the unit normals to S_1, S_2, and Γ, respectively and point outwards from those surfaces. Invoking the continuity of the temperature and normal heat flux at S_1, S_2 readily shows that the first and third integrals in (4.3) cancel each other, and so do the second and fifth integral. Thus, the expression in (4.3) reduces to

$$J=\int_{\Gamma} q_i^{(2)}n_i^{(\Gamma)}\phi'^{(2)}d\Gamma, \tag{4.4}$$

which is the left-hand side of (4.1). On the other hand, the quantity J, as given in (4.2), can be also transformed into

$$J=-\int_{V_1} k_{ij}^{(1)}\phi_{,j}^{(1)}\phi'^{(1)}_{,i}dV_1-\int_{V_0} k_{ij}^{(0)}\phi_{,j}^{(0)}\phi'^{(0)}_{,i}dV_0-\int_{V_2} k_{ij}^{(2)}\phi_{,j}^{(2)}\phi'^{(2)}_{,i}dV_2 \tag{4.5}$$

where we have used the fact that the heat flux is divergenceless, and invoked also the constitutive laws of the phases characterized by symmetric conductivity tensor $k_{ij}^{(r)}$,which we have assumed to be anisotropic for the sake of generality. Thus J, as given by (4.4) and (4.5), is symmetric in the unprimed and primed fields and this concludes the proof.

In the next two subsections we consider the situation in which the interphase residing in volume V_0 is represented either by Model I or Model II which are of $O(t)$ accuracy, and prove that the reciprocal theorem (4.1) continues to be fulfilled in their presence within $O(t)$ accuracy.

4.1. Model I and the reciprocal theorem

We now refer to Fig. 2b in which the geometry of the interphase is left intact, and on S_1 and S_2 prevail the conditions (3.12) and (3.13) which govern Model I. The heat fluxes are divergences in the volumes V_1 and V_2, but this time one is not allowed to invoke the fields within the interphase V_0. In similarity to (4.2), the following quantity is now defined:

$$K=\int_{V_1}(q_i^{(1)}\phi'^{(1)})_{,i}dV_1+\int_{V_2}(q_i^{(2)}\phi'^{(2)})_{,i}dV_2, \tag{4.6}$$

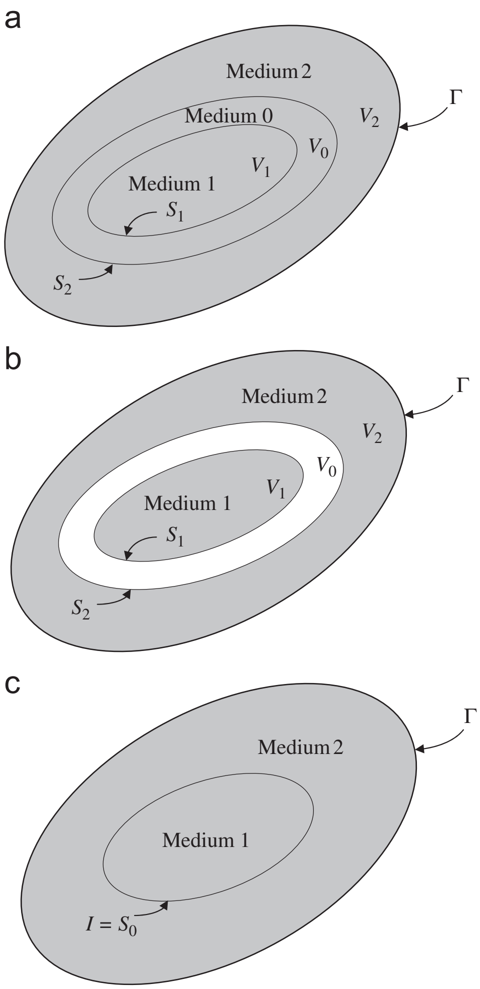

Fig. 2. The configurations used in Section 4 concerning the reciprocal theorem: (a) the three-phase configuration, (b) the configuration for Model I, (c) the configuration for Model II.

which transforms into

$$K=\int_{S_1} q_i^{(1)} n_i^{(S_1)} \phi'^{(1)} dS_1 + \int_{\Gamma} q_i^{(2)} n_i^{(\Gamma)} \phi'^{(2)} d\Gamma - \int_{S_2} q_i^{(2)} n_i^{(S_2)} \phi'^{(2)} dS_2, \tag{4.7}$$

and can be rewritten as

$$\int_{\Gamma} q_i^{(2)} n_i^{(\Gamma)} \phi'^{(2)} d\Gamma = K-E, \tag{4.8}$$

where E has been defined by

$$E=\int_{S_2} q_n^{(2)} \phi'^{(2)} dS_2 - \int_{S_1} q_n^{(1)} \phi'^{(1)} dS_1. \tag{4.9}$$

Following the same development which has led to (4.5), it is first concluded from (4.6) that the quantity K is symmetric in the unprimed and primed fields since the conductivity tensors of the adjacent media are given by $k_{ij}^{(r)}=k_r\delta_{ij}$. It follows therefore that the left hand side of (4.8) will be symmetric in the unprimed and primed fields if E can be shown to possess that property. The proof to the symmetry of E under the conditions (3.12) and (3.13) will be given in this subsection.

A transformation which is central in achieving the proof is the introduction of new variables $U,\ W,\ Q_n, P_n$ as follows:

$$U=(\phi^{(2)})_{t/2}+(\phi^{(1)})_{-t/2}, \quad W=(\phi^{(2)})_{t/2}-(\phi^{(1)})_{-t/2},$$
$$Q_n=(q_n^{(2)})_{t/2}+(q_n^{(1)})_{-t/2}, \quad P_n=(q_n^{(2)})_{t/2}-(q_n^{(1)})_{-t/2}, \tag{4.10}$$

resulting in

$$(\phi^{(2)})_{h/2} = (U+W)/2, \quad (\phi^{(2)})_{-h/2} = (U-W)/2,$$
$$(q_n^{(2)})_{h/2} = (Q_n+P_n)/2, \quad (q_n^{(2)})_{-h/2} = (Q_n-P_n)/2. \tag{4.11}$$

With the new variables of (4.10), Eqs. (3.12) and (3.13) transform now into

$$W = \left(-\frac{t}{2k_0(v_1,v_2)}\right)Q_n + O(t^2), \tag{4.12}$$

$$P_n = \left(\frac{t}{2}\right)\left(\frac{1}{R_1^{(S_0)}} + \frac{1}{R_2^{(S_0)}}\right)Q_n + \left(\frac{t}{2}\right)\left\{\mathrm{div}_{S_0}\left(k_0(v_1,v_2)\mathbf{grad}_{S_0}U\right)\right\} + O(t^2), \tag{4.13}$$

or

$$Q_n = \left(-\frac{2k_0(v_1,v_2)}{t}\right)W + O(t), \tag{4.14}$$

$$P_n = -\left(\frac{1}{R_1^{(S_0)}} + \frac{1}{R_2^{(S_0)}}\right)k_0(v_1,v_2)W + \left(\frac{t}{2}\right)\left\{\mathrm{div}_{S_0}\left(k_0(v_1,v_2)\mathbf{grad}_{S_0}U\right)\right\} + O(t^2). \tag{4.15}$$

where it should be recalled that in view of their definitions in (4.10) and the representations in (3.12) and (3.13), W and P_n are of $O(t)$.

We now return to (4.9), and substitute in it (2.12) and (4.11) to obtain

$$E = \int_{S_0}\left(\frac{Q_n+P_n}{2}\right)\left(\frac{U'+W'}{2}\right)\left(1-\frac{(t/2)}{R_1^{(S_0)}}\right)\left(1-\frac{(t/2)}{R_2^{(S_0)}}\right)dS_0 - \int_{S_0}\left(\frac{Q_n-P_n}{2}\right)\left(\frac{U'-W'}{2}\right)\left(1+\frac{(t/2)}{R_1^{(S_0)}}\right)\left(1+\frac{(t/2)}{R_2^{(S_0)}}\right)dS_0 \tag{4.16}$$

where the integrals are now over the surface S_0. Moreover, invoking the fact that W, W', P_n, P_n' are each of $O(t)$, and keeping *only* the $O(t)$ terms provides after some algebra

$$E = (1/4)\int_{S_0} P_n(2U')dS_0 + (1/4)\int_{S_0} Q_n\left[2W' - t\left(\frac{1}{R_1^{(S_0)}} + \frac{1}{R_2^{(S_0)}}\right)U'\right]dS_0 + O(t^2). \tag{4.17}$$

Next, substituting the expressions of (4.14) and (4.15) for P_n and Q_n into (4.17) yields

$$E = (t/4)\int_{S_0}\left\{\mathrm{div}_{S_0}\left(k_0(v_1,v_2)\mathbf{grad}_{S_0}U\right)\right\}U'dS_0 - \int_{S_0} k_0(v_1,v_2)\frac{WW'}{t}dS_0 + O(t^2). \tag{4.18}$$

In order to prove the symmetry of the first integral in the above equation we now transform it as

$$\int_{S_0}\left\{\mathrm{div}_{S_0}\left(k_0(v_1,v_2)\mathbf{grad}_{S_0}U\right)\right\}U'dS_0 = -\int_{S_0} k_0(v_1,v_2)\left\{\mathbf{grad}_{S_0}U \cdot \mathbf{grad}_{S_0}U'\right\}dS_0, \tag{4.19}$$

whose validity can be observed by letting first

$$dS_0 = h_1^{(S_0)}h_2^{(S_0)}dv_1dv_2, \tag{4.20}$$

then making use of the relations in (2.23) for $\mathrm{div}_{S_0}\left(k_0(v_1,v_2)\mathbf{grad}_{S_0}U\right)$, followed performing by integrations by parts separately with respect to the variables v_1 andv_2 (see also Appendix 3 in Van Bladel (2006) for similar surface integral theorems). Finally, substitution of (4.19) in (4.18) provides

$$E = -(t/4)\int_{S_0} k_0(v_1,v_2)\left\{\mathbf{grad}_{S_0}U \cdot \mathbf{grad}_{S_0}U'\right\}dS_0 - \int_{S_0} k_0(v_1,v_2)\frac{WW'}{t}dS_0 + O(t^2). \tag{4.21}$$

which is symmetric in the unprimed and primed fields.

To conclude, having shown the symmetry of both E and K, it follows from (4.8) that the left hand-side of (4.1) is also symmetric in the unprimed and primed fields, and thus equal to its right hand-side. The validity of the reciprocal theorem (4.1) in the presence of the interphase representation by Model I has thus been proved.

4.2. Model II and the reciprocal theorem

We now refer to Fig. 2c in which the interphase has been removed and replaced by an interface on which prevail the conditions (3.21) and (3.22) which govern Model II. A similar development to that existing between (4.1) and (4.9) shows that Model II will fulfill the reciprocal theorem if the following quantity consisting of surface integrals on the interface I can be proved to be symmetric in the unprimed and primed fields:

$$\hat{E} = \int_{S_I}(q_n^{(2)})_+(\phi'^{(2)})_+dS_I - \int_{S_I}(q_n^{(1)})_-(\phi'^{(1)})_-dS_I. \tag{4.22}$$

As in the previous subsection, we now introduce the new field variables

$$\hat{U}=(\phi^{(2)})_+ + (\phi^{(1)})_-, \quad \hat{W}=(\phi^{(2)})_+ - (\phi^{(1)})_-,$$
$$\hat{Q}_n=(q_n^{(2)})_+ + (q_n^{(1)})_-, \quad \hat{P}_n=(q_n^{(2)})_+ - (q_n^{(1)})_-, \tag{4.23}$$

which imply

$$(\phi^{(2)})_+ = (\hat{U}+\hat{W})/2, \quad (\phi^{(1)})_- = (\hat{U}-\hat{W})/2,$$
$$(q_n^{(2)})_+ = (\hat{Q}_n+\hat{P}_n)/2, \quad (q_n^{(1)})_- = (\hat{Q}_n-\hat{P}_n)/2, \tag{4.24}$$

and cast Eqs. (3.21) and (3.22) of Model II in the following forms:

$$\hat{W}=-\left(\frac{t}{2}\right)\left\{\frac{f_2+f_1}{2}\hat{Q}_n+\frac{f_2-f_1}{2}\hat{P}_n\right\}+O(t^2), \tag{4.25}$$

$$\hat{P}_n=\left(\frac{t}{2}\right)\left\{\mathrm{div}_{S_I}\left(k_0(v_1,v_2)\mathbf{grad}_{S_I}\hat{U}\right)-k_2\Delta_{S_I}\left((\hat{U}+\hat{W})/2\right)-k_1\Delta_{S_I}\left((\hat{U}-\hat{W})/2\right)\right\}+O(t^2), \tag{4.26}$$

where we have defined

$$f_1=\frac{1}{k_0}-\frac{1}{k_1}, \quad f_2=\frac{1}{k_0}-\frac{1}{k_2}. \tag{4.27}$$

However, noting that $\hat{W}$ and $\hat{P}_n$ are each of $O(t)$, (4.25) and (4.26) can be further transformed within $O(t)$ accuracy into

$$\hat{W}=-\left(\frac{t}{2}\right)\left\{\frac{f_2+f_1}{2}\hat{Q}_n\right\}+O(t^2)\Leftrightarrow\hat{Q}_n=-4\left\{\frac{1}{t(f_2+f_1)}\right\}\hat{W}+O(t), \tag{4.28}$$

$$\hat{P}_n=\left(\frac{t}{2}\right)\left\{\mathrm{div}_{S_I}\left(k_0(v_1,v_2)\mathbf{grad}_{S_I}\hat{U}\right)-[(k_2+k_1)/2]\Delta_{S_I}\hat{U}\right\}+O(t^2). \tag{4.29}$$

Now, we return to the proof of the symmetry of (4.22). Substitution of (4.24) in (4.22) leads to

$$\hat{E}=\int_{S_I}\left(\frac{\hat{Q}_n+\hat{P}_n}{2}\right)\left(\frac{\hat{U}'+\hat{W}'}{2}\right)dS_I-\int_{S_I}\left(\frac{\hat{Q}_n-\hat{P}_n}{2}\right)\left(\frac{\hat{U}'-\hat{W}'}{2}\right)dS_I. \tag{4.30}$$

Recalling that $\hat{W}$, $\hat{W}'$, $\hat{P}_n$, $\hat{P}_n{}'$ are each of $O(t)$, the above equation is transformed within $O(t)$ accuracy into

$$\hat{E}=(1/4)\int_{S_I}\hat{P}_n(2\hat{U}')dS_I+(1/4)\int_{S_I}\hat{Q}_n(2\hat{W}')dS_I+O(t^2). \tag{4.31}$$

Next, substitution of $\hat{Q}_n$ and $\hat{P}_n$ from (4.28) and (4.29) into (4.31) provides

$$\hat{E}=(1/4)\int_{S_I}t\left\{\mathrm{div}_{S_I}\left(k_0(v_1,v_2)\mathbf{grad}_{S_I}\hat{U}\right)\hat{U}'\right\}dS_I$$
$$-(1/8)\int_{S_I}\left\{t(k_2+k_1)\left(\Delta_{S_I}\hat{U}\right)\hat{U}'\right\}dS_I-\int_{S_I}\left\{\frac{1}{t(f_1+f_2)}\right\}(2\hat{W}\hat{W}')dS_I+O(t^2). \tag{4.32}$$

Finally, use of the integral transformations

$$\int_{S_I}\{\mathrm{div}_{S_I}(k_0(v_1,v_2)\mathbf{grad}_{S_I}U)\}U'dS_I=-\int_{S_I}k_0(v_1,v_2)\{\mathbf{grad}_{S_I}U\cdot\mathbf{grad}_{S_I}U'\}dS_I,$$
$$\int_{S_I}\{(\Delta_{S_I}U)U'\}dS_I=-\int_{S_I}\{\mathbf{grad}_{S_I}U\cdot\mathbf{grad}_{S_I}U'\}dS_I, \tag{4.33}$$

casts (4.32) into the following form

$$\hat{E}=-(1/4)\int_{S_I}t\left\{k_0(v_1,v_2)\left(\mathbf{grad}_{S_I}\hat{U}\cdot\mathbf{grad}_{S_I}\hat{U}'\right)\right\}dS_I$$
$$+(1/8)\int_{S_I}\left\{t(k_2+k_1)\left(\mathbf{grad}_{S_I}\hat{U}\cdot\mathbf{grad}_{S_I}\hat{U}'\right)\right\}dS_I-\int_{S_I}\left\{\frac{2\hat{W}\hat{W}'}{t(f_1+f_2)}\right\}dS_I+O(t^2), \tag{4.34}$$

which is symmetric in the unprimed and primed fields. Thus, the validity of the reciprocal theorem in the presence of Model II has been proved.

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Appendix A

In a recent study by Sussmann et al. (2011), the two-dimensional version of Model II, in a setting limited to a circular interphase of constant conductivity, was incorporated in a finite-element formulation. In this Appendix we revisit part of the analysis existing in that work in regard to the standing of Model II vis-à-vis the reciprocal theorem, and modify it in the light of the contents of Section 4 of the present study.

Eqs. (29) and (33) in Sussmann et al. (2011) characterize Model II in two-dimensional conduction for the case of a circular interphase which we rewrite here using the notation of the present paper

$$(\phi^{(2)})_+-(\phi^{(1)})_-=-\left(\frac{t}{2}\right)\left[\left(\frac{1}{k_0}-(1-\alpha)\frac{1}{k_2}\right)(q_n)_+ +\left(\frac{1}{k_0}-(1+\alpha)\frac{1}{k_2}\right)(q_n)_-\right]+O(t^2), \tag{A.1}$$

$$(q_n^{(2)})_+-(q_n^{(1)})_-=\left(\frac{t}{2(1+\gamma)}\right)\left[\left(\frac{k_0-(1-\alpha)k_2}{R_I^2}\right)\left(\frac{\partial^2\phi^{(2)}}{\partial\theta^2}\right)_+ +\left(\frac{k_0-(1+\alpha)k_1}{R_I^2}\right)\left(\frac{\partial^2\phi^{(1)}}{\partial\theta^2}\right)_-\right]+O(t^2), \tag{A.2}$$

where $\gamma=\frac{\alpha t}{2R_I}$, θ denotes the angular coordinate in polar coordinates, and α was a parameter which was introduced therein which fixed the position of the interface I in regard to the interphase geometry. The radius of the interface I was related to the inner and outer circles and α by the relation

$$R_I=\frac{R_1+R_2}{2}+\frac{\alpha t}{2}. \tag{A.3}$$

If the location of the interface I coincides with the mid-circle then $\alpha=0$, and Eqs. (A.1) and (A.2) become a special case our Eqs. (3.21) and (3.22) in Section 3 of the present paper. The motivation in Sussmann et al. (2011) for allowing a flexibility in the location of the interface I was with an effort to render Model II self-adjoint, which is equivalent to demanding its fulfillment of the reciprocal theorem. Yet, with the help of the transformation scheme (4.23) and (4.24) introduced here, it can be shown that self-adjointness was sought in Sussmann et al. (2011) without omitting certain terms which were of higher order than $O(t)$. In this Appendix we will show that by omitting those terms, the self-adjointness of Model II will indeed be forthcoming for any value of the parameter α.

Solving Eqs. (A.1) and (A.2) for $(q_n^{(2)})_+$ and $(q_n^{(1)})_-$ provided Eqs. (40) and (41) in Sussmann et al. (2011) which are rewritten below:

$$(q_n^{(2)})_+=\frac{(\phi^{(2)})_+-(\phi^{(1)})_-+BC(\phi^{(2)}_{,ss})_++BD(\phi^{(1)}_{,ss})_-}{A+B},$$
$$(q_n^{(1)})_-=\frac{(\phi^{(2)})_+-(\phi^{(1)})_--AC(\phi^{(2)}_{,ss})_+-AD(\phi^{(1)}_{,ss})_-}{A+B}, \tag{A.4}$$

where the second derivative with respect to the arc-length coordinate of the circle I is denoted by $(\ )_{,ss}$ and the constants $A,\ B,\ C,\ D$ are defined by

$$A=-\left(\frac{t}{2}\right)\left[\frac{1}{k_0}-(1-\alpha)\frac{1}{k_2}\right],\quad B=-\left(\frac{t}{2}\right)\left[\frac{1}{k_0}-(1+\alpha)\frac{1}{k_1}\right],$$
$$C=\left(\frac{t}{2(1+\gamma)}\right)[k_0-(1-\alpha)k_2],\quad D=\left(\frac{t}{2(1+\gamma)}\right)[k_0-(1+\alpha)k_1]. \tag{A.5}$$

If we substitute now the expressions of (A.4) into (4.22) we get

$$\begin{aligned}\hat{E}&=\int_{S_I}\frac{[(\phi^{(2)})_+-(\phi^{(1)})_-][(\phi'^{(2)})_+-(\phi'^{(1)})_-]}{A+B}ds_I\\&\quad+\int_{S_I}\frac{BC(\phi^{(2)}_{,ss})_+(\phi'^{(2)})_++BD(\phi^{(1)}_{,ss})_-(\phi'^{(2)})_++AC(\phi^{(2)}_{,ss})_+(\phi'^{(1)})_-+AD(\phi^{(1)}_{,ss})_-(\phi'^{(1)})_-}{A+B}ds_I\\&=\int_{S_I}\frac{[(\phi^{(2)})_+-(\phi^{(1)})_-][(\phi'^{(2)})_+-(\phi'^{(1)})_-]}{A+B}ds_I\\&\quad-\int_{S_I}\frac{BC(\phi^{(2)}_{,s})_+(\phi'^{(2)}_{,s})_++BD(\phi^{(1)}_{,s})_-(\phi'^{(2)}_{,s})_++AC(\phi^{(2)}_{,s})_+(\phi'^{(1)}_{,s})_-+AD(\phi^{(1)}_{,s})_-(\phi'^{(1)}_{,s})_-}{A+B}ds_I,\end{aligned} \tag{A.6}$$

where ds_I is the infinitesimal arc-length coordinate along I, and integration by parts has been performed. It is seen from the above expression that the first integral is symmetric in the primed and unprimed fields, whereas one is tempted to state that the second integral could be made symmetric if the following demand is made

$$BD=AC. \tag{A.7}$$

In Sussmann et al. (2011) this condition fixed the value of α in terms of the material parameters of the problem in Eq. (63) therein. However, in view of the transformations (4.23) and (4.24) of the present paper, it becomes now apparent that self adjointness to $O(t)$ accuracy could indeed have been achieved in that work without introducing the demand (A.7).

In order to see this, let us substitute (4.24) into (A.6) to obtain

$$\hat{E}=\int_{S_I}\frac{\hat{W}\hat{W}'}{A+B}ds_I-\frac{1}{4(A+B)}\int_{S_I}[(\hat{U}'_{,s}+\hat{W}'_{s})(BC)(\hat{U}_{,s}+\hat{W}_{,s})+(\hat{U}'_{,s}+\hat{W}'_{,s})(BD)(\hat{U}_{,s}-\hat{W}_{,s})]ds_I$$
$$-\frac{1}{4(A+B)}\int_{S_I}[(\hat{U}'_{,s}-\hat{W}'_{,s})(AC)(\hat{U}_{,s}+\hat{W}_{,s})+(\hat{U}'_{,s}-\hat{W}'_{,s})(AD)(\hat{U}_{,s}-\hat{W}_{,s})]ds_I. \tag{A.8}$$

In order to simplify the above equation within $O(t)$ accuracy we note that $A+B$ is of $O(t)$, and (BC), (BD), (AC), (AD) are each of $O(t^2)$. Moreover from (4.24) and (A.1), we also note that $\hat{W}$ and $\hat{W}'$ are of $O(t)$. Thus, the part of (A.8) which is of $O(t)$ is

$$\hat{E}=\int_{S_I}\frac{\hat{W}\hat{W}'}{A+B}ds_I-\frac{1}{4(A+B)}\int_{S_I}\hat{U}'_s\hat{U}_{,s}(BC+BD+AC+AD)ds_I. \tag{A.9}$$

Thus, $\hat{E}$ is symmetric in the unprimed and primed fields, and Model II becomes self-adjoint within $O(t)$ accuracy, no matter where the interface I is located with respect to the interphase geometry.

Finally, in order to make contact with the general result of (4.34), we now let $\alpha=0$, for which one has

$$A+B=-\left(\frac{t}{2}\right)\left(\frac{2}{k_0}-\frac{1}{k_1}-\frac{1}{k_2}\right),\quad \frac{BC+BD+AC+AD}{4(A+B)}=(t/8)\left[2k_0-k_1-k_2\right], \tag{A10}$$

so as to provide from (A.9), the following result:

$$\hat{E}=-\left(\frac{t}{2}\right)^{-1}\left(\frac{2}{k_0}-\frac{1}{k_1}-\frac{1}{k_2}\right)^{-1}\int_{S_I}\hat{W}\hat{W}'ds_I-(t/8)\left[2k_0-k_1-k_2\right]\int_{S_I}\hat{U}'_s\hat{U}_{,s}ds_I+O(t^2). \tag{A.11}$$

It can be now readily verified that the general result (4.34) of the present paper reduces to (A.11) for the special case of the two-dimensional setting. Thus, the consistency of the presently modified derivation of Sussmann et al. (2011) with the general result of (4.34) has been established.

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